

Classical Simulation of Limited-Width Cluster-State Quantum Computation

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We present a classical protocol, using the matrix product-state representation, to simulate cluster-state quantum computation at a cost polynomial in the number of qubits in the cluster and exponential in d —the width of the cluster. We use this result to show that any log-depth quantum computation in the gate array model, with gates linking only nearby qubits, can be simulated efficiently on a classical computer.

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The model of cluster-state quantum computation proposed by Raussendorf and Briegel [1] has attracted much interest in recent years. This model demonstrates some remarkable features, most notably the fact that once a particular multiqubit state (the cluster state) has been prepared, the whole processing of the data is implemented by measurements of individual qubits and feed forward (that is, the measurement carried out on one qubit may depend on the outcomes of previous measurements). In general, the cluster state is thought of as a two-dimensional grid of qubits, entangled by applying controlled-phase (CPHASE) operations between neighboring pairs. Due to this entangled structure, any quantum circuit can be simulated by cluster-state computation [1] with a polynomial resource overhead (in terms of the number of qubits and elementary operations). Yet, the computational power of a cluster does not depend only on the amount of resources that were invested in its construction but also on its geometry. It was shown by Nielsen that any computation implemented with a linear cluster (a single chain of qubits) can be simulated efficiently on a classical computer [2].

In this Letter, we analyze the computational power of limited-width cluster states. We show that any computation on a cluster state in the shape of a rectangular grid can be simulated on a classical computer at a cost which is quadratic in the number of qubits and polynomial in 2^d , where d is the width of the cluster. Therefore, if we limit the width so it scales like the log of the number of logical qubits, then any computation on that cluster can be simulated efficiently. We will generalize our proof to include also clusters in which the connections between qubits are not necessarily between nearest neighbors but are bounded by some constant length.

Since any quantum computation can be implemented by cluster-state computation, our results imply that any quantum computation in the gate array model with depth that scales like the log of the number of qubits and where the range of the interactions is bounded by a constant (that is, two-qubit gates are applied only to qubits which are not too far apart) can be efficiently simulated on a classical computer. Similar results concerning the computational power of limited-depth quantum gate arrays have been

reported recently [3] following from a completely different approach.

Our approach relies on the representation of the cluster state as a matrix product state (MPS). We shall follow here the construction of Vidal [4], through which the matrix representation of the state of n qubits ($|\Psi\rangle$) is obtained by a sequence of $n - 1$ Schmidt decompositions. These Schmidt decompositions relate to a certain ordering of the qubits, where the k th decomposition corresponds to a partition of the system into the first k qubits and the remaining $n - k$ qubits. Let us first write the state of the system in the computational basis

$$|\Psi\rangle = \sum_{i_1, \dots, i_n=0}^1 C_{i_1, \dots, i_n} |i_1\rangle \cdots |i_n\rangle. \quad (1)$$

The key point of the construction is the representation of the coefficients C_{i_1, \dots, i_n} as a product of n tensors ($\Gamma^{[k]}$) and $n - 1$ vectors ($\lambda^{[k]}$)

$$C_{i_1, \dots, i_n} = \sum_{\alpha_1, \dots, \alpha_n} \Gamma_{\alpha_1}^{[1]i_1} \lambda_{\alpha_1}^{[1]} \Gamma_{\alpha_1 \alpha_2}^{[2]i_2} \lambda_{\alpha_2}^{[2]} \Gamma_{\alpha_2 \alpha_3}^{[3]i_3} \cdots \Gamma_{\alpha_{n-1}}^{[n]i_n}. \quad (2)$$

Each index α_k goes from 1 to the number of terms in the k th Schmidt decomposition (the Schmidt number), and the elements of the vector $\lambda^{[k]}$ are the corresponding Schmidt coefficients. Note that the Schmidt number (or the log of this number) relating to a partition of the system can be seen as a measure of the entanglement between the two parts [5] and, as such, cannot increase under local operations and classical communication. We denote the maximal Schmidt number over all $n - 1$ Schmidt decompositions by χ .

Let us now discuss in brief Vidal's construction process. One starts by expressing $|\Psi\rangle$ using the first Schmidt decomposition (corresponding to the partition into qubit 1 and the rest):

$$|\Psi\rangle = \sum_{\alpha_1} \lambda_{\alpha_1}^{[1]} |\Phi_{\alpha_1}^{[1]}\rangle |\Phi_{\alpha_1}^{[2 \cdots n]}\rangle. \quad (3)$$

Expressing the Schmidt vector $|\Phi_{\alpha_1}^{[1]}\rangle$ in the computational basis, we obtain

$$|\Psi\rangle = \sum_{i_1\alpha_1} \Gamma_{\alpha_1}^{[1]i_1} \lambda_{\alpha_1}^{[1]} |i_1\rangle |\Phi_{\alpha_1}^{[2\cdots n]}\rangle. \quad (4)$$

In the next step, the states $|\Phi_{\alpha_1}^{[2\cdots n]}\rangle$ are expressed in terms of the computational basis states of the second qubit and of the Schmidt vectors $|\Phi_{\alpha_2}^{[3\cdots n]}\rangle$ corresponding to the second Schmidt decomposition, to obtain

$$|\Psi\rangle = \sum_{i_1\alpha_1} \sum_{i_2\alpha_2} \Gamma_{\alpha_1}^{[1]i_1} \lambda_{\alpha_1}^{[1]} \Gamma_{\alpha_1\alpha_2}^{[2]i_2} \lambda_{\alpha_2}^{[2]} |i_1\rangle |i_2\rangle |\Phi_{\alpha_1}^{[3\cdots n]}\rangle. \quad (5)$$

This process can be repeated qubit by qubit until the representation in (2) is obtained. The crucial point here is that one can always express the Schmidt vectors $|\Phi_{\alpha_{k-1}}^{[k\cdots n]}\rangle$ of qubits $[k, \dots, n]$, obtained by the $(k-1)$ th Schmidt decomposition, in terms of the Schmidt vectors $|\Phi_{\alpha_k}^{[k+1\cdots n]}\rangle$, obtained by the k th Schmidt decomposition. Hence, we can always write

$$|\Phi_{\alpha_{k-1}}^{[k\cdots n]}\rangle = \sum_{i_k\alpha_k} \Gamma_{\alpha_{k-1}\alpha_k}^{[k]i_k} \lambda_{\alpha_k}^{[k]} |i_k\rangle |\Phi_{\alpha_k}^{[k+1\cdots n]}\rangle. \quad (6)$$

It is fairly easy to see that if this was not the case and one of $|\Phi_{\alpha_{k-1}}^{[k\cdots n]}\rangle$ had a component outside the subspace spanned by the states $|i_k\rangle |\Phi_{\alpha_k}^{[k+1\cdots n]}\rangle$, then the overall state $|\Psi\rangle$ could not lie within the subspace spanned by $|\Phi_{\alpha_k}^{[1\cdots k]}\rangle |\Phi_{\alpha_k}^{[k+1\cdots n]}\rangle$, in contradiction to the k th Schmidt decomposition.

A description of the state of our system in terms of the $\Gamma^{[k]}$'s and $\lambda^{[k]}$'s would require approximately $(2\chi^2 + \chi)n$ parameters, instead of the 2^n coefficients $(C_{i_1\cdots i_n})$ required to represent the state in the computational basis. The parameter which determines the size of the description of the system is therefore χ . In general, χ is of order 2^n and the MPS representation is not very useful. However, if the state does not carry much entanglement, then χ may be smaller and the MPS representation may be advantageous. In particular, if χ scales like $\text{poly}(n)$, then one would have an efficient description of the state involving only $\text{poly}(n)$ parameters.

Since cluster-state computation involves only single-qubit operations, it is easy to see that the Schmidt number associated with any partition cannot increase and, hence, that χ will not increase during the computation. Therefore, if the initial cluster state has an efficient MPS representation, all later states can also be represented efficiently. Furthermore, we show that these later representations can be obtained efficiently and, hence, prove the following: Any computation consisting of (projective) single-qubit measurements and feed forward on a system of n qubits, where the maximal Schmidt number for all bipartitions of the system along a certain ordering is χ , can be classically simulated at a cost of $O(n^2 \text{poly}(\chi))$ in computational time and memory space.

In order to simulate a single-qubit measurement, we need to calculate the probabilities for the two outcomes, sample from the probability distribution, project the state

of the system accordingly, and then renormalize the projected state. An arbitrary single-qubit measurement can be implemented by first applying a single-qubit unitary and then measuring the qubit in the computational basis. Let us examine the effect of a single-qubit unitary acting on the k th qubit on the MPS representation of the state. Clearly, each of the computational basis states would undergo the following transformation:

$$|i_k\rangle \rightarrow \sum_{i'_k} U_{i'_k i_k} |i'_k\rangle. \quad (7)$$

This operation can be incorporated into the tensors corresponding to the k qubit— $\Gamma_{\alpha_k\alpha_{k+1}}^{[k]i_k}$ —by replacing them with

$$\tilde{\Gamma}_{\alpha_k\alpha_{k+1}}^{[k]i_k} = U_{0i_k} \Gamma_{\alpha_k\alpha_{k+1}}^{[k]0} + U_{1i_k} \Gamma_{\alpha_k\alpha_{k+1}}^{[k]1}. \quad (8)$$

Updating the MPS representation after a single-qubit unitary would therefore take $O(\chi^2)$ basic operations.

The probabilities of the outcomes in the computational basis can be easily calculated from the following representation of the state of the system, obtained by using the $(k-1)$ th Schmidt decomposition, and Eq. (6):

$$\sum_{i_k} \sum_{\alpha_{k-1}\alpha_k} \lambda_{\alpha_{k-1}}^{[k-1]i_k} \Gamma_{\alpha_{k-1}\alpha_k}^{[k]i_k} \lambda_{\alpha_k}^{[k]} |\Phi_{\alpha_{k-1}}^{[1\cdots k-1]}\rangle |i_k\rangle |\Phi_{\alpha_k}^{[k+1\cdots n]}\rangle. \quad (9)$$

The probability $p(i_k)$ for receiving the outcome i_k in a measurement of the k qubit is therefore obtained from the tensor $\Gamma_{\alpha_{k-1}\alpha_k}^{[k]i_k}$ and the vectors $\lambda_{\alpha_{k-1}}^{[k-1]}$ and $\lambda_{\alpha_k}^{[k]}$. Defining

$$A_{\alpha_{k-1}\alpha_k}^{i_k} = \lambda_{\alpha_{k-1}}^{[k-1]i_k} \Gamma_{\alpha_{k-1}\alpha_k}^{[k]i_k} \lambda_{\alpha_k}^{[k]} \quad (10)$$

and using the orthogonality of the Schmidt vectors, we have

$$p(i_k) = \sum_{\alpha_{k-1}\alpha_k} |A_{\alpha_{k-1}\alpha_k}^{i_k}|^2. \quad (11)$$

Sampling from this probability distribution, and receiving outcome $|r_k\rangle$, the state of the system after projection and renormalization will be $|r_k\rangle |\Psi'\rangle$, where

$$|\Psi'\rangle = \frac{1}{\sqrt{p(r_k)}} \sum_{\alpha_{k-1}\alpha_k} A_{\alpha_{k-1}\alpha_k}^{r_k} |\Phi_{\alpha_{k-1}}^{[1\cdots k-1]}\rangle |\Phi_{\alpha_k}^{[k+1\cdots n]}\rangle. \quad (12)$$

In what follows, we leave out the state of the measured qubits (which remain in a product state with the rest of the system) and consider the MPS representation of the remaining qubits. This representation must now be updated, since the λ 's and Γ 's above do not correspond to Schmidt decompositions of $|\Psi'\rangle$, and, in order to be able to calculate the probability distribution for the measurement of the remaining qubits efficiently, the correct MPS representation must be recovered. Indeed, as we consider a general n -qubit state (where the measurement of one qubit might affect all other qubits), all of the Γ 's and λ 's must be updated. Tracing over qubits $1, \dots, k-1$, we obtain the χ by χ reduced density matrix $\rho^{[k+1\cdots n]}$

$$\rho_{\alpha_k \alpha'_k}^{[k+1 \dots n]} = \frac{1}{p(r_k)} \sum_{\alpha_{k-1}} A_{\alpha_{k-1} \alpha_k}^{r_k} (A_{\alpha_{k-1} \alpha'_k}^{r_k})^*. \quad (13)$$

Given the reduced density matrix, we can find its eigenvalues $\tilde{\lambda}_{\beta_{k-1}}^{[k-1]}$, which are the Schmidt coefficients for the above partition. We can also find its eigenvectors $M_{\beta_{k-1} \alpha_k}$, which represent the new Schmidt vectors $|\Phi_{\beta_{k-1}}^{[k+1 \dots n]}\rangle$ in the basis of the old Schmidt vectors:

$$|\Phi_{\beta_{k-1}}^{[k+1 \dots n]}\rangle = \sum_{\alpha_k} M_{\beta_{k-1} \alpha_k} |\Phi_{\alpha_k}^{[k+1 \dots n]}\rangle. \quad (14)$$

To calculate the Schmidt coefficients ($\tilde{\lambda}_{\beta_k}^{[k]}$) for the next partition, between qubits $[1, \dots, k-1, k+1]$ and $[k+1, \dots, n]$, we write the state of system as follows:

$$\begin{aligned} |\Psi'\rangle &= \sum_{\beta_{k-1}} \lambda_{\beta_{k-1}}^{[k-1]} |\Phi_{\beta_{k-1}}^{[1 \dots k-1]}\rangle |\Phi_{\beta_{k-1}}^{[k+1 \dots n]}\rangle \\ &= \sum_{\beta_{k-1} \alpha_k} \lambda_{\beta_{k-1}}^{[k-1]} M_{\beta_{k-1} \alpha_k} |\Phi_{\beta_{k-1}}^{[1 \dots k-1]}\rangle |\Phi_{\alpha_k}^{[k+1 \dots n]}\rangle \\ &= \sum_{\substack{\beta_{k-1} \alpha_{k+1} \\ i_{k+1}}} B_{\beta_{k-1} \alpha_{k+1}}^{i_{k+1}} |\Phi_{\beta_{k-1}}^{[1 \dots k-1]}\rangle |i_{k+1}\rangle |\Phi_{\alpha_{k+1}}^{[k+2 \dots n]}\rangle, \end{aligned} \quad (15)$$

where

$$B_{\beta_{k-1} \alpha_{k+1}}^{i_{k+1}} = \sum_{\alpha_k} \lambda_{\beta_{k-1}}^{[k-1]} M_{\beta_{k-1} \alpha_k} \Gamma_{\alpha_k \alpha_{k+1}}^{[k+1] i_{k+1}} \lambda_{\alpha_{k+1}}^{[k+1]}. \quad (16)$$

We can now write the reduced density matrix of qubits $[k+2, \dots, n]$ in terms of the tensor B

$$\rho_{\alpha_{k+1} \alpha'_{k+1}}^{[k+2 \dots n]} = \sum_{\beta_{k-1} i_{k+1}} B_{\beta_{k-1} \alpha_{k+1}}^{i_{k+1}} (B_{\beta_{k-1} \alpha'_{k+1}}^{i_{k+1}})^*. \quad (17)$$

Having calculated $\rho^{[k+2 \dots n]}$, we can now find its eigenvalues $\tilde{\lambda}_{\beta_k}^{[k]}$ (the Schmidt coefficients) and its eigenvectors $M_{\beta_k \alpha_{k+1}}$, which represent the new Schmidt vectors $|\Phi_{\beta_k}^{[k+2 \dots n]}\rangle$ in the basis of the old Schmidt vectors $|\Phi_{\alpha_{k+1}}^{[k+2 \dots n]}\rangle$, as in (14).

The relation between $|\Phi_{\beta_k}^{[k+2 \dots n]}\rangle$ and $|\Phi_{\beta_{k-1}}^{[k+1 \dots n]}\rangle$ defines the new tensors $\tilde{\Gamma}_{\beta_{k-1} \beta_k}^{[k+1] i_{k+1}}$ corresponding to qubit $k+1$. Examining (6), we see that

$$\begin{aligned} \tilde{\Gamma}_{\beta_{k-1} \beta_k}^{[k+1] i_{k+1}} &= \frac{1}{\tilde{\lambda}_{\beta_k}^{[k]}} \langle i_{k+1}, \Phi_{\beta_k}^{[k+2 \dots n]} | \Phi_{\beta_{k-1}}^{[k+1 \dots n]} \rangle \\ &= \frac{1}{\tilde{\lambda}_{\beta_k}^{[k]}} \sum_{\alpha_k \alpha_{k+1}} M_{\beta_k \alpha_{k+1}}^* M_{\beta_{k-1} \alpha_k} \Gamma_{\alpha_k \alpha_{k+1}}^{[k+1] i_{k+1}} \lambda_{\alpha_{k+1}}^{[k+1]}, \end{aligned} \quad (18)$$

where we have used (14) and the corresponding relation for $|\Phi_{\beta_k}^{[k+2 \dots n]}\rangle$ in the last line.

We can proceed in the same manner to obtain all the reduced density matrices in one direction ($\rho^{[k+3 \dots n]}$ to $\rho^{[n]}$) and their eigenvalues and eigenvectors. At each step, the

reduced density matrix is given as a function of the eigenvectors and eigenvalues obtained in the previous step (as well as the old Γ 's and λ 's). The new $\tilde{\Gamma}$'s are found using the eigenvectors of two consecutive steps as in (18). In order to update the MPS representation in the other direction, we first express $|\Phi_{\beta_{k-1}}^{[1 \dots k-1]}\rangle$ in terms of $|\Phi_{\alpha_{k-1}}^{[1 \dots k-2]}\rangle$ using (12) and (14) and

$$|\Phi_{\beta_{k-1}}^{[1 \dots k-1]}\rangle = \frac{1}{\tilde{\lambda}_{\beta_{k-1}}} \langle \Phi_{\beta_{k-1}}^{[k+1 \dots n]} | \Psi' \rangle, \quad (19)$$

then reexpress these states in terms of $|\Phi_{\alpha_{k-2}}^{[1 \dots k-2]}\rangle |i_{k-1}\rangle$ using

$$|\Phi_{\alpha_{k-1}}^{[1 \dots k-1]}\rangle = \sum_{i_{k-1} \alpha_{k-2}} \lambda_{\alpha_{k-2}}^{[k-2]} \Gamma_{\alpha_{k-2} \alpha_{k-1}}^{[k-1] i_{k-1}} |\Phi_{\alpha_{k-2}}^{[1 \dots k-2]}\rangle |i_{k-1}\rangle. \quad (20)$$

Expanding $|\Psi'\rangle$ as in (15) using (19) and (20), and tracing over qubits $[k-1, k+1, \dots, n]$, we obtain $\rho^{[1 \dots k-2]}$. The eigenvalues and eigenvectors of this matrix can be used to obtain $\tilde{\lambda}_{\beta_{k-2}}^{[k-2]}$ and $\tilde{\Gamma}_{\beta_{k-2} \beta_{k-1}}^{[k-1] i_{k-1}}$ in a similar way as before, and, repeating this procedure, we can obtain all of the remaining $\tilde{\Gamma}$'s and $\tilde{\lambda}$'s.

In each step of the updating procedure, we deal with a constant number of (at most) χ by χ matrices, requiring $\text{poly}(\chi)$ basic operations, and hence updating the whole state after a measurement requires $n \text{poly}(\chi)$ basic operations. As we can apply at most n single-qubit measurements to the state, the overall computation can be simulated with $O(n^2 \text{poly}(\chi))$ computational resources. Note that we do not include the cost of computing the feed forward from the measurement results, as this is common to both the quantum computation and the classical simulation (and for standard cluster-state computation can be computed efficiently).

In the above procedure, we simulate the single-qubit measurements in the same order as they are measured in the actual computation. This is usually independent of the order in which qubits in the MPS representation are numbered, which we choose so as to minimize χ . However, if we can number the MPS representation in the same order as the qubits are measured without significantly increasing χ , then the simulation can be considerably simplified. In this case, we do not have to update the MPS representation after the measurements, as the probability distribution can be calculated directly from the projected state. The cost of the simulation in this case would be $O(n\chi^2)$.

Let us now consider a cluster state in the shape of a rectangular grid of width d and length $l > d$ where each qubit is entangled to all its nearest neighbors. In order to construct our MPS representation, we choose the following ordering of the qubits: We start from the qubit in the top-left corner and number the qubits column by column until we reach the qubit in the bottom-right corner. In this case, the maximal Schmidt number χ for all bipartitions of the

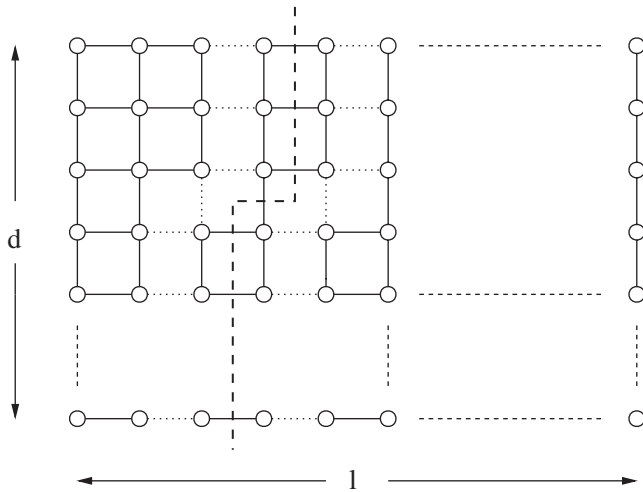


FIG. 1. A rectangular cluster state. The solid lines represent entanglement (CPHASE connections), the thick dashed line shows a typical partition of the cluster, and the dotted lines show the connections that do not exist in the incomplete cluster.

system is 2^d . To prove this, we consider a typical bipartition and an incomplete cluster, similar to the original but where the $2d$ qubits next to the partition are only entangled with each other (and not with the rest of the cluster), as shown in Fig. 1. The original cluster can be constructed from the incomplete cluster by local unitary operations (that is, operations which act on only one side of the partition) and, therefore, both have the same Schmidt number S with respect to this partition. The state of the incomplete cluster can be written as $|\Psi_A\rangle|P\rangle|\Psi_B\rangle$, where $|P\rangle$ stands for the entangled state of the $2d$ qubits next to the partition and $|\Psi_{A/B}\rangle$ are the states of the remaining qubits on the left and right sides, respectively. Since $|P\rangle$ contains only $2d$ qubits, it is clear that $S \leq 2^d$. In fact, one can easily verify that $S = 2^d$, since $|P\rangle$ itself can be constructed from d maximally entangled pairs of qubits by local unitary operations. As no bipartition has a greater Schmidt number than S , we have $\chi = 2^d$.

In general, we expect both d and l to scale as $\text{poly}(N)$, where N is the number of logical qubits, and, hence, the above simulation would require exponential resources. However, any cluster-state computation implemented by a grid of physical qubits with limited width, that is, where d scales like (at most) $\log(N)$, can be simulated at a polynomial cost in time and memory space.

We can also extend our approach to more general cluster states, in which non-neighboring qubits are connected by CPHASE operations, as long as these connections have limited range. Consider a cluster where the vertical distance between connected qubits (across the width d) is not limited, and the horizontal distance is limited by r . That is, a qubit in column k may be connected to any qubit in

columns $\{k - r, \dots, k + r\}$. Looking at a typical partition where the left side consists of the first $k - 1$ columns and part of the k th column, and the right side contains the rest of the cluster, we consider an incomplete cluster where the block consisting of columns $\{k - r, \dots, k + r\}$ is isolated from the rest of the cluster (i.e., in the incomplete cluster, the qubits of this block have the same connections between themselves but no connections to the rest of the cluster). As in the previous case, the state of the incomplete cluster is given by $|\Psi_A\rangle|P'\rangle|\Psi_B\rangle$, where $|P'\rangle$ is the state of the isolated block. $|P'\rangle$ consists of only $(2r + 1)d$ qubits, so the Schmidt number for the partition must be less than $2^{(r+1/2)d}$. As before, the original cluster can be recovered by local CPHASE operations; hence, $\chi \leq 2^{(r+1/2)d}$. It is therefore clear that any cluster-state computation on a rectangular grid can be efficiently simulated with a classical computer as long as either d (the width of the cluster) or r (the range of the connections) scales like $\log(N)$ and the other is constant.

In our general procedure, we did not specify to which qubits the input is introduced and in which order the measurements are performed in the computation. Given an efficient MPS representation of the initial state of the cluster, any computation can be simulated efficiently. Thus, considering the $l \times d$ rectangular grid above, we can allocate a column of physical qubits for each logical qubit (measuring the physical qubits, say, from top to bottom). Simulating a quantum gate array in this way, l would be proportional to the number of logical qubits N , d would be proportional to the depth of the computation (i.e., the number of time steps), and r would be proportional to the range over which gates can act. Therefore, we can also state that any quantum computation in the gate array model where either the depth of the computation or the range of the interaction scales like $\log(N)$, while the other is bounded by a constant, can be efficiently simulated on a classical computer.

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