Moduli Space of Non-Abelian Vortices

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(Received 17 November 2005; published 27 April 2006)

We completely determine the moduli space $\mathcal{M}_{N,k}$ of k vortices in $U(N)$ gauge theory with N Higgs fields in the fundamental representation. Its open subset for separated vortices is found as the symmetric product $(\mathbf{C}\times\mathbf{C}P^{N-1})^k/\mathfrak{S}_k$. Orbifold singularities of this space correspond to coincident vortices and are resolved resulting in a smooth moduli manifold. The relation to Kähler quotient construction is discussed.

DOI: 10.1103/PhysRevLett.96.161601 PACS numbers: 11.27.+d, 11.10.Lm, 11.25.-w, 11.30.Pb

Introduction.—Vortices are very important solitons in various areas of physics [1]: high energy physics, cosmology, condensed matter physics, and nuclear physics. Vortices in Abelian gauge theory have been well studied so far [2–4]. Recently, vortices in non-Abelian gauge theory (called non-Abelian vortices) have attracted much attention [5–7] because a monopole is confined in the Higgs phase with non-Abelian vortices attached as a dual picture of quark confinement [8] (see also [9,10] for related models). It is very important to determine the moduli space of vortices. It describes the vortex scattering in $d = 2 + 1$ [4], is used for the reconnection of vortex (cosmic) strings in $d = 3 + 1$ [11,12], and is needed for the vortex counting in $d = 1 + 1$, similarly to the instanton counting. Identifying vortices with certain D branes in a D-brane configuration in string theory, the Kähler quotient construction of the moduli space of non-Abelian vortices was suggested [5]. We have determined the moduli space of domain walls [13] and other solitons [14] by introducing the method of the moduli matrix. In this Letter we completely determine the moduli space of non-Abelian vortices by applying this method.

Vortex equations and their solutions.—We consider vortex solutions in $d = 3, 4, 5, 6$. Field contents are a gauge field W_M ($M = 0, 1, \ldots, d-1$), two $N\times N$ matrices H^1 and H^2 of Higgs fields and adjoint scalars Σ^I ($I = 1, \ldots, 6-d$). The Lagrangian in $d = 6$ is

$$\mathcal{L}_6 = \mathrm{Tr}\left[-\frac{1}{2g^2}F_{MN}F^{MN} + \mathcal{D}^M H^i(\mathcal{D}_M H^i)^\dagger\right] - V, \quad (1)$$

with $V = \frac{g^2}{4}\,\mathrm{Tr}[(H^1H^{1\dagger} - H^2H^{2\dagger} - c\mathbf{1}_{N_C})^2 + 4H^2H^{1\dagger}H^1H^{2\dagger}]$, where the triplet of Fayet-Iliopoulos parameters is chosen to the third direction $(0, 0, c > 0)$. This Lagrangian enjoys $U(N)$ gauge symmetry as well as $SU(N)$ flavor symmetry. By adding fermions this Lagrangian becomes supersymmetric with eight supercharges. The Lagrangian in $d = 3, 4, 5$ is obtained by trivial dimensional reductions, in which the adjoint scalars Σ^I appear from higher dimensional components of the gauge field. The scalars Σ^I trivially vanish in vortex solutions and we do not need them. In either dimension, the vacuum is the so-called color-flavor locking phase, $H^1 = \sqrt{c}\mathbf{1}_N$ and $H^2 = 0$ where the symmetry of the Lagrangian is broken to $SU(N)_{\mathrm{G+F}}$. This symmetry will be further broken in the presence of vortices and therefore acts as an isometry on the moduli space.

In the following we simply set $H^2 = 0$ and $H \equiv H^1$. The Bogomolnyi completion leads to the vortex equations

$$0 = \mathcal{D}_1 H + i\mathcal{D}_2 H, \quad 0 = F_{12} + \frac{g^2}{2}(c\mathbf{1}_N - HH^\dagger), \quad (2)$$

for vortices in the x^1-x^2 plane and their tension

$$T = -c\int d^2x\,\mathrm{Tr}F_{12} = 2\pi ck, \quad (3)$$

with $k(\in \mathbf{Z}_{\geq 0})$ measuring the winding number of the $U(1)$ part of the broken $U(N)$ gauge symmetry.

Defining a complex coordinate $z \equiv x^1 + ix^2$, the first vortex Eq. (2) can be solved as

$$H = S^{-1}H_0(z), \qquad W_1 + iW_2 = -i2S^{-1}\bar{\partial}_z S, \quad (4)$$

with $S = S(z, \bar{z}) \in GL(N, \mathbf{C})$ defined by the second Eq. (4), and $H_0(z)$ an arbitrary N by N matrix holomorphic with respect to z, which we call the *moduli matrix*. With a gauge invariant quantity $\Omega \equiv SS^\dagger$ the second vortex Eq. (2) can be rewritten as

$$\partial_z(\Omega^{-1}\bar{\partial}_z\Omega) = \frac{g^2}{4}(c\mathbf{1}_N - \Omega^{-1}H_0H_0^\dagger). \quad (5)$$

We call this the *master equation* for vortices [15]. This equation is expected to give no additional moduli parameters. It was proved for the $U(1)$ case [3] and is consistent with the index theorem [5] in general N as seen below.

Equation (5) implies asymptotic behavior $\Omega \to \frac{1}{c}H_0H_0^\dagger$ for $z \to \infty$. Then the tension (3) can be rewritten as

$$T = 2\pi ck = -i\frac{c}{2}\oint dz\,\partial\log(\det H_0) + \mathrm{c.c.} \quad (6)$$

We thus obtain the boundary condition on S^1_∞ for H_0 as $\det(H_0) \sim z^k$ for $z \to \infty$. Since any point at infinity S^1_∞ must belong to the same gauge equivalence class, elements in H_0 must be polynomial functions of z. (If exponential factors exist they become dominant at boundary S^1_∞ and the configuration fails to converge to the same gauge equivalence class there.) From the expression (6), we find that $\det H_0(z)$ has k zeros at $z = z_i$, which can be defined as the positions of vortices: $\det H_0(z_i) = 0$.

There exists a redundancy in the solution (4): physical quantities H and $W_{1,2}$ are invariant under the "V transformation"

$$H_0 \to VH_0, \qquad S \to VS, \qquad \det V = \text{const} \neq 0, \quad (7)$$

with $V = V(z) \in GL(N, \mathbf{C})$, whose elements are holomorphic with respect to z. Here, the third condition is necessary to maintain the vortex number k unchanged. The moduli space $\mathcal{M}_{N,k}$ for k vortices in $U(N)$ gauge theory can be formally expressed as a quotient

$$\mathcal{M}_{k,N} = \frac{\{H_0(z) | H_0(z) \in M_N, \text{deg}\det[H_0(z)] = k\}}{\{V(z) | V(z) \in M_N, \det V(z) = \text{const} \neq 0\}}, \quad (8)$$

where M_N denotes a set of holomorphic $N \times N$ matrices and "deg" denotes a degree of polynomials.

The moduli space of vortices.—The V transformation (7) allows us to reduce degrees of polynomials in H_0 by applying the division algorithm. After fixing the V transformation completely, any moduli matrix H_0 is uniquely transformed to a triangular matrix, which we call the standard form,

$$H_0 = \begin{pmatrix} P_1(z) & R_{2,1}(z) & R_{3,1}(z) & \cdots & R_{N,1}(z) \\ 0 & P_2(z) & R_{3,2}(z) & \cdots & R_{N,2}(z) \\ \vdots & & \ddots & & \vdots \\ & & & & R_{N,N-1}(z) \\ 0 & \cdots & & 0 & P_N(z) \end{pmatrix} \quad (9)$$

with the monic polynomial $P_r(z) = \prod_{i=1}^{k_r}(z - z_{r,i})$ and $R_{r,m}(z) \in \text{Pol}(z; k_r)$. Here $\text{Pol}(z; n)$ denotes a set of polynomial functions of order less than n. The standard form (9) has *one-to-one correspondence* to a point in the moduli space. Since $\det(H_0) = \prod_{r=1}^{N} P_r(z) \sim z^k$ asymptotically for $z \to \infty$, we obtain the vortex number $k = \sum_{r=1}^{N} k_r$ from Eq. (6) and realize the positions of the k vortices as the zeros of $P_r(z)$. Collecting all matrices with given k in the standard form (9) we obtain the whole moduli space $\mathcal{M}_{N,k}$ for k vortices. Its generic points are parameterized by the matrix with $k_N = k$ and $k_r = 0$ for $r \neq N$,

$$H_0 = \begin{pmatrix} \mathbf{1}_{N-1} & -\vec{R}(z) \\ 0 & P(z) \end{pmatrix}, \quad (10)$$

where $P(z) = \prod_{i=1}^{k}(z - z_i)$ and $[\vec{R}(z)]^r = R_r(z) \in \text{Pol}(z; k)$ is an $N - 1$ vector. This moduli matrix contains the maximal number of the moduli parameters. The dimension of the moduli space is $\dim(\mathcal{M}_{N,k}) = 2kN$ coinciding with the index theorem [5].

The standard form (9) has the merit of covering the entire moduli space only once without any overlap. However, we should parameterize the moduli space with overlapping patches to clarify the global structure of the moduli space. We can parameterize the moduli space by a set of ${}_{k+N-1}C_k$ patches defined by

$$(H_0)^r_s = z^{k_s}\delta^r_s - T^r_s(z), \qquad T^r_s(z) \in \text{Pol}(z; k_s). \quad (11)$$

Coefficients of monomials in ${T^r}_s(z)$ are moduli parameters as coordinates in a patch. We denote this patch by $\mathcal{U}^{(k_1,k_2,\ldots,k_N)}$. We can show that each patch fixes the V transformation (7) completely including any discrete subgroup, and therefore that the isomorphism $\mathcal{U}^{(k_1,k_2,\ldots,k_N)} \simeq \mathbf{C}^{kN}$ holds. The transition functions between these patches are given by the V transformation (7), completely defining the moduli space as a smooth manifold, $\mathcal{M}_{N,k} \simeq \bigcup \mathcal{U}^{(k_1,k_2,\ldots,k_N)}$.

To see this explicitly we show an example of one vortex ($k = 1$). In this case there exist N patches

$$\begin{pmatrix} 1 & & 0 & -b_1^{(N)} \\ & \ddots & & \vdots \\ 0 & & 1 & -b_{N-1}^{(N)} \\ 0 & \ldots & 0 & z - z_0 \end{pmatrix} \simeq \begin{pmatrix} 1 & & -b_1^{(N-1)} & 0 \\ & \ddots & \vdots & \\ 0 & & z - z_0 & 0 \\ 0 & \ldots & -b_N^{(N-1)} & 1 \end{pmatrix} \simeq \cdots. \quad (12)$$

Transition functions among these patches are given by the V equivalence (7) as $(b_1^{(N)}, \ldots, b_{N-1}^{(N)}, 1) = b_{N-1}^{(N)}(b_1^{(N-1)}, \ldots, b_{N-2}^{(N-1)}, 1, b_N^{(N-1)}) = \cdots = b_1^{(N)}(1, b_2^{(1)}, \ldots, b_{N-1}^{(1)}, b_N^{(1)})$. These b's are the standard patches for $\mathbf{C}P^{N-1}$ and are called orientational moduli. We thus have $\mathcal{M}_{N,k=1} \simeq \mathbf{C} \times \mathbf{C}P^{N-1}$ recovering the result [6] obtained by a symmetry argument.

Properties of the moduli space.—We have found that zeros of $P_r(z)$ in Eq. (9) are the positions of the vortices. We will clarify the meaning of the remaining moduli parameters $R_{r,m}(z)$ in Eq. (9) from now on. For simplicity we consider the patch $\mathcal{U}^{(0,\ldots,0,k)}$ given in Eq. (10) and study $\vec{R}(z)$ therein. To this end, we shall introduce the basis $\{e^i(z)\}$ (i=1,2,...,k) of the space of polynomial $\text{Pol}(z; k)$. For example, the simplest complete basis is the monomial basis $e^i_{\text{m}}(z) \equiv z^{i-1}$. Elements of $\text{Pol}(z; k)$ can be expressed by coefficients of monomials in that basis. In terms of vortex positions z_j given in the polynomial $P(z) = \prod_{i=1}^{k}(z - z_i)$ with degree k in Eq. (10), we define another basis called *point basis* (Lagrange interpolation coefficient)

$$e^i_{\text{p}}(z) \equiv \prod_{j=1,(i\neq j)}^{k}\left(\frac{z - z_j}{z_i - z_j}\right), \qquad e^i_{\text{p}}(z_j) = \delta^i_j. \quad (13)$$

The point basis is defined only when $z_i \neq z_j$ for $i \neq j$, namely, for the separated vortices. Elements in $\text{Pol}(z; k)$ can be expressed by values at different k points $\{z_i\}$ in this basis. For example, $\vec{R}(z)$ in (10) can be expressed as $\vec{R}(z) = \sum_{i=1}^{k} \vec{b}_i e^i_{\text{p}}(z)$ with $\vec{b}_i \equiv \vec{R}(z_i)$. Notice that the k by k matrix U in $e^i_{\text{p}}(z) = \sum_{n=1}^{k} U^i_n e^n_{\text{m}}(z)$ gives the Vandermonde determinant $\det U^{-1} = \prod_{k \geq j > i \geq 1}(z_j - z_i)$, ensuring the completeness of the point basis (13). We thus find one-to-one correspondence between $\vec{b}_i$ and $\vec{R}(z)$.

Now we are ready to understand the physical meaning of the moduli parameters in $\vec{R}(z)$. To this end, we consider the infinitesimal $SU(N)$ isometry with an element

$$u(\xi) = \begin{pmatrix} \mathbf{0}_{N-1} & -\vec{\xi} \\ \vec{\xi}^{\dagger} & 0 \end{pmatrix}$$

($\vec{\xi}$ is an $N-1$ vector) acting on H_0 as

$$\delta H_0(z) = v(\xi, z)H_0(z) + H_0(z)u(\xi), \quad (14)$$

with an infinitesimal V transformation (7) $v(\xi, z)$ needed to pull back to (10). This leads to

$$\delta\vec{R}(z) = \vec{\xi} + \vec{R}(z)[\vec{\xi}^\dagger \cdot \vec{R}(z)] + \vec{s}_{\xi^\dagger}(z)P(z). \quad (15)$$

Here $\vec{s}_{\xi^\dagger}(z)$ is a polynomial function for the pull back which is uniquely determined for $\vec{R}$ to be in $\mathrm{Pol}(z;k)$ again. Noting $P(z_i) = 0$ $(i = 1, \ldots, k)$ we obtain $\vec{b}_i = \vec{R}(z_i)$ as $\delta\vec{b}_i = \vec{\xi} + \vec{b}_i(\vec{\xi}^\dagger \cdot \vec{b}_i)$ by setting $z = z_i$ in (15). This is precisely the $SU(N)$ transformation law for $\mathbf{C}P^{N-1}$. Namely, a set of $(z_i, \vec{b}_i)$ parameterizes $\mathbf{C}\times\mathbf{C}P^{N-1}$, like the moduli of the single vortex mentioned above [16]. Taking into account the fact that H_0 approaches to the one in (12) for a single vortex with the orientational moduli $\vec{b}_i$ in the vicinity of the ith vortex, with $|z - z_i| \ll |z - z_j|$ for all $j(\neq i)$ holding, we thus find the asymptotic form (open set) of the moduli space for separated vortices,

$$\mathcal{M}_{N,k} \leftarrow (\mathbf{C}\times\mathbf{C}P^{N-1})^k/\mathfrak{S}_k \equiv S^k(\mathbf{C}\times\mathbf{C}P^{N-1}) \quad (16)$$

with the $\mathfrak{S}_k$ permutation group exchanging the positions of the vortices [17]. Here $\leftarrow$ denotes a map to resolve the singularities on the right-hand side. Equation (16) can be easily expected from physical intuition; for instance the $k = 2$ case was found in [12]. The most important thing is how orbifold singularities of the right-hand side in (16) are resolved by coincident vortices, which we explain below. In the $N = 1$ case, $\mathcal{M}_{N=1,k} \simeq \mathbf{C}^k/\mathfrak{S}_k$ holds instead of (16) [3].

Relation to the Kähler quotient.—Next we investigate the relation between our moduli space and that from the Kähler quotient [5] mainly in the patch $\mathcal{U}^{(0,\ldots,0,k)}$. For that purpose, it is important to introduce a surjective map from the space of polynomials $\mathrm{Pol}(z)$ to $\mathrm{Pol}(z;k)$ by

$$q(z) = r(z) + s(z)P(z) = r(z) \mod P(z), \quad (17)$$

with $q(z), s(z) \in \mathrm{Pol}(z)$ and $r(z) \in \mathrm{Pol}(z;k)$. The last equality in (17) gives a map from $q(z)$ to $r(z)$ by modulo $P(z)$. We can extract the moduli parameters from $P(z)$ and $\vec{R}(z)$ as constant matrices $\mathbf{Z}$ and $\mathbf{\Psi}$:

$$ze^i(z) \equiv (\mathbf{Z})^i_j e^j(z) \mod P(z), \quad (18)$$

$$\begin{pmatrix} \vec{R}(z) \\ 1 \end{pmatrix} \equiv (\mathbf{\Psi})_i e^i(z). \quad (19)$$

When we change the basis as $e'^i(z) = U^i_j e^j(z)$ by $U \in GL(k, \mathbf{C})$, these matrices transform as $\mathbf{Z}' = U\mathbf{Z}U^{-1}$, $\mathbf{\Psi}' = \mathbf{\Psi}U^{-1}$. This is precisely the complexified gauge transformation appearing in the Kähler quotient construction [5] in which the moduli space is given by k by k matrix Z and N by k matrix ψ. The concrete correspondence is obtained by fixing the imaginary part of the gauge transformation as $\mathcal{M}_{N,k} \simeq \{\mathbf{Z}, \mathbf{\Psi}\}//GL(k, \mathbf{C}) \simeq \{(Z, \psi)|[Z^\dagger, Z] + \psi^\dagger\psi \propto \mathbf{1}_k\}/U(k)$.

For the separated vortices, the point basis (13) gives us $\mathbf{\Psi}$ for the orientational moduli and the diagonal matrix $\mathbf{Z}$ whose elements correspond to the positions of the vortices

$$\mathbf{Z} = \mathrm{diag}(z_1, z_2, \ldots, z_k), \qquad \mathbf{\Psi} = \begin{pmatrix} \vec{b}_1 & \cdots & \vec{b}_k \\ 1 & \cdots & 1 \end{pmatrix}. \quad (20)$$

As we have mentioned above, the point basis (13) cannot be used for coincident vortices, $z_i = z_j$ for $i \neq j$. We can deal with them by noting differentiations at z_i naturally arise in the limit $z_j \to z_i$. Let us assume that d_I vortices coincide at $z = z_I$, and divide the labels i to distinguish vortices as $\{i\} = \{(I, \alpha_I)\}$ with $\alpha_I = 1, \ldots, d_I$. We define the *generalized point basis* by

$$e_{\mathrm{p}}^{(I,\alpha_I)}(z) \equiv \sum_{n=1}^{k} U^{(I,\alpha_I)}{}_n e_{\mathrm{m}}^n(z), \quad (21)$$

$$\frac{1}{(\alpha_I - 1)!}\frac{d^{\alpha_I-1}e_{\mathrm{p}}^{(J,\alpha_J)}(z)}{dz^{\alpha_I-1}}\bigg|_{z=z_I} = \delta^I_J\delta^{\alpha_J}{}_{\alpha_I}, \quad (22)$$

where U is a k by k invertible matrix, whose inverse and determinant are given by

$$(U^{-1})^m{}_{(I,\alpha_I)} = {}_{m-1}C_{\alpha_I-1}z_I^{m-\alpha_I}, \quad (23)$$

$$\det U^{-1} = \prod_I\prod_{J>I}(z_J - z_I)^{d_Jd_I}, \quad (24)$$

respectively. In this basis any function can be expressed by a set of differentiations at $z = z_I$. When no vortices coincide, $d_I = 1$ for all I, the generalized point basis (21) reduces to the point basis (13). The matrix $\mathbf{Z}$ in the basis (21) takes the Jordan normal form

$$\mathbf{Z}^{(I,\alpha_I)}{}_{(J,\beta_J)} = \delta^I_J(\mathbf{z}_I)^{\alpha_I}{}_{\beta_J}, \mathbf{z}_I = \begin{pmatrix} z_I & 1 & & 0 \\ 0 & z_I & \ddots & \\ \vdots & & \ddots & 1 \\ 0 & \cdots & 0 & z_I \end{pmatrix} \quad (25)$$

and $(\Psi)_{(I,\alpha_I)} = (\Psi_I)_{\alpha_I}$ is given by

$$\mathbf{\Psi}_I = \begin{pmatrix} \vec{R}(z_I) & \vec{R}'(z_I) & \cdots & \frac{1}{(d_I-1)!}\partial_z^{d_I-1}\vec{R}(z_I) \\ 1 & 0 & \cdots & 0 \end{pmatrix}. \quad (26)$$

Emergence of the Jordan matrix $\mathbf{Z}$ is analogous to instantons in terms of the Hilbert scheme [18].

So far in this section, we have dealt with the only patch $\mathcal{U}^{(0,\ldots,0,k)}$ to show correspondence between our construction and the Kähler quotient construction. In order to complete the correspondence, we have to verify it over whole region of the moduli space. In what follows we illustrate the correspondence in the case of $(N, k) = (2, 2)$. The moduli space $\mathcal{M}_{N=2,k=2}$ is parameterized by the three patches $\mathcal{U}^{(0,2)}$, $\mathcal{U}^{(1,1)}$, $\mathcal{U}^{(2,0)}$ defined in H_0's

$$\begin{pmatrix} 1 & -az - b \\ 0 & z^2 - \alpha z - \beta \end{pmatrix}, \begin{pmatrix} z - \phi & -\varphi \\ -\tilde{\varphi} & z - \tilde{\phi} \end{pmatrix}, \begin{pmatrix} z^2 - \alpha z - \beta & 0 \\ -a'z - b' & 1 \end{pmatrix},$$

respectively. The moduli data in these patches can be summarized by two matrices $\mathbf{Z}$ and $\mathbf{\Psi}$ as follows

$$\{\mathbf{Z}, \mathbf{\Psi}\} = \left\{\begin{pmatrix} 0 & 1 \\ \beta & \alpha \end{pmatrix}, \begin{pmatrix} b & a \\ 1 & 0 \end{pmatrix}\right\}, \left\{\begin{pmatrix} \phi & \varphi \\ \tilde{\varphi} & \tilde{\phi} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 0 & 1 \\ \beta & \alpha \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ b' & a' \end{pmatrix}\right\}. \quad (27)$$

The first one corresponds to the matrices $\{\mathbf{Z}, \mathbf{\Psi}\}$ in Eqs. (18) and (19) in the monomial basis. The V transformation (7) between these three patches can be expressed by the complexified gauge transformation between moduli data as $(\mathbf{Z}', \mathbf{\Psi}') = (U\mathbf{Z}U^{-1}, \mathbf{\Psi}U^{-1})$ with appropriate $U \in GL(2, \mathbf{C})$.

In conclusion we have determined the moduli space of non-Abelian vortices in $U(N)$ gauge theory with N Higgs fields in the fundamental representation. The orbifold singularity appearing in the asymptotic form (16) of separated vortices is correctly resolved in the full moduli space, resulting a complete smooth manifold. The relation between our moduli space and the one proposed in the D-brane technique is explicitly shown in the case of $N = k = 2$. The complete identification for general (N, k) is an important future work. By solving the master Eq. (5) numerically we should be able to calculate the moduli metric. Refining the discussion of reconnection of non-Abelian cosmic string [12] using the moduli metric is to be explored. We also leave analysis of semilocal vortices in $U(N_\mathrm{C})$ gauge theory with $N_\mathrm{F}(>N_\mathrm{C})$ flavors as a future problem. Another interesting extension is studying non-Abelian vortices on Riemann surfaces [19].

M. E., M. N., and K. O. (K. O.) would like to thank Nick Manton (Kenichi Konishi) for a useful discussion and are grateful for hospitality at DAMTP. This work is supported in part by Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology, Japan No. 13640269 (N. S.) and 16028203 for the priority area ''origin of mass'' (N. S.). The work of M. N. and K. O. (M. E. and Y. I.) is supported by Japan Society for the Promotion of Science under the Postdoctoral (Predoctoral) Research Program. M. N. thanks Hiraku Nakajima for a comment on Ref. [17].

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[15] The master equation reduces to the so-called Taubes equation [3] in the $N = 1$ case by rewriting $c\Omega(z, \bar{z}) = |H_0|^2 e^{-\xi(z,\bar{z})}$ with $H_0 = \prod_i (z - z_i)$. Note that $\log\Omega$ is regular everywhere while ξ is singular at vortex points.

[16] In the patch $\mathcal{U}^{(0,\ldots,0,k)}$, all vortices are aligned in $\mathbf{C}P^{N-1}$'s when $b_i = 0$ for all i, and b_i describe fluctuations from that configuration.

[17] This is a half of the moduli space of k separated $U(N)$ instantons on noncommutative $\mathbf{R}^4$, $(\mathbf{C}^2 \times T^* \mathbf{C}P^{N-1})^k / \mathfrak{S}_k$.

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[19] J. M. Baptista, Commun. Math. Phys. **261**, 161 (2006).