Hawking Radiation from Charged Black Holes via Gauge and Gravitational Anomalies

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Extending the method of Robinson and Wolczek, we show that in order to avoid a breakdown of general covariance and gauge invariance at the quantum level the total flux of charge and energy in each outgoing partial wave of a charged quantum field in a Reissner-Nordström black hole background must be equal to that of a $(1 + 1)$-dimensional blackbody at the Hawking temperature with the appropriate chemical potential.

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Introduction.—There are several derivations of Hawking radiation. Hawking's original one [1,2], which calculates the Bogoliubov coefficients between in and out states for a body collapsing to form a black holes, is the most direct. An elegant derivation based on Euclidean quantum gravity [3] has been interpreted as a calculation of tunneling through classically forbidden trajectories [4]. It remains of interest to consider alternative derivations, since each is based on different assumptions, which might or might not be incorporated within a complete theory of quantum gravity.

Recently, Robinson and Wilczek proposed a new partial derivation of Hawking radiation [5], which ties its existence to the cancellation of gravitational anomalies at the horizon. As explained there, this derivation has important advantages: it localizes the source of the anomaly at the horizon, where the geometry is nonsingular yet the equations simplify; it ties in with the effective field theory approach to the membrane paradigm [6,7]; and the validity of anomalies seems a particularly reliable assumption, since anomalies are a profound feature of quantum field theory. In this Letter we extend the analysis of [5] to Hawking radiation of charged particles from Reissner-Nordström (RN) black holes. To do this we will need to consider gauge anomalies at the horizon in addition to gravitational anomalies.

To identify the ground state of a quantum field, one normally associates positive energy with occupation of modes of positive frequency. But in defining positive frequency, one must refer to a specific definition of time. In the Boulware state, vacuum is defined in terms of the Schwarzschild time. That is a natural definition of time in the exterior region, but it becomes problematic at the horizon. In this state freely falling observers feel a singular flux of energy-momentum tensor as they pass through the horizon. That unphysical behavior arises from nontrivial occupation of modes that propagate nearly along the horizon at high frequency. The Unruh vacuum [8], in contrast, uses a time variable adapted to the near-horizon geometry to define energy. It associates (large) positive energy to the offending modes, so they are unoccupied. The basic idea in [5] is to form an effective theory that integrates out the offending modes. Having excluded propagation along one lightlike direction, the effective near-horizon quantum field then becomes chiral, and contains gravitational anomalies. The original underlying theory is generally covariant, so failure of the effective theory to reflect this symmetry should be relieved by introducing an extra effect. It was shown that the energy-momentum flux associated with Hawking radiation emanating from the horizon cancels the anomaly at the horizon.

Setting.—Consider the partial wave decomposition of a charged complex field in a RN black hole background. It can be either a charged scalar or a Dirac fermion. The gauge potential ($A_t = -\phi$) and the metric of the spacetime are given by

$$A = -\frac{Q}{r}dt, \tag{1}$$

$$ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2 d\Omega^2_{(d-2)}. \tag{2}$$

$d^2\Omega_{(d-2)}$ is the line element on the $(d-2)$ sphere and $f(r)$ is

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = \frac{(r-r_+)(r-r_-)}{r^2}, \tag{3}$$

where $r_\pm = M \pm \sqrt{M^2 - Q^2}$ are the radii of the outer and inner horizons. The surface gravity at the outer horizon is given by $\kappa \equiv \frac{1}{2}(\partial_r f)|_{r_+}$.

Upon transforming to the r_* "tortoise" coordinate defined by $\frac{\partial r}{\partial r_*} = f(r)$ and performing a partial wave decomposition, one finds that the effective radial potentials for partial wave modes of the field contain the factor $f[r(r_*)]$ and vanish exponentially fast near the horizon. The same applies to mass terms or other interactions. Thus physics near the horizon can be described using an infinite collection of massless $(1+1)$-dimensional fields, each partial wave propagating in a spacetime with a metric given by the "$r-t$" section of the full spacetime metric (2). This is a kind of dimensional reduction from d dimensions to $d=2$. We exploit that simplification. Two points should be noted. First, the effective two-dimensional current or energy-momentum tensor are given by integrating the d-dimensional ones over a $(d-2)$ sphere. In the four-dimensional case, for example, $J^\mu_{(2)} = \int r^2 d\Omega^2 J^\mu_{(4)} = 4\pi r^2 J^\mu_{(4)}(r)$. Second, when reducing to $d=2$, the Lagrangian contains an r^2 factor, which cannot be interpreted as the two-dimensional metric. This factor can be interpreted as a dilaton background coupled to the charged fields [9,10].

Since the horizon is a null hypersurface, modes interior to the horizon can not affect physics outside the horizon, *classically*. If we formally remove modes to obtain the effective action in the exterior region, it becomes anomalous with respect to gauge or diffeomorphism symmetries. The underlying theory is, of course, invariant. Therefore those anomalies must be cancelled by quantum effects of the modes that were irrelevant classically. In the following we show that the conditions for anomaly cancellation at the horizon are met by the Hawking flux of charge and energy momentum.

Gauge anomaly.—First we investigate the charged current and gauge anomaly at the horizon. The effective theory outside of the horizon r_+ is defined in the region $r \in [r_+, \infty]$. If we first omit the ingoing modes in the region $r \in [r_+, r_+ + \epsilon]$ near the horizon, the gauge current exhibits an anomaly there. The consistent form of $d=2$ Abelian anomaly (for review, see [11,12]) is given by

$$\nabla_\mu J^\mu = \pm \frac{e^2}{4\pi\sqrt{-g}} \epsilon^{\mu\nu} \partial_\mu A_\nu, \tag{4}$$

where $+(-)$ corresponds to left(right)-handed fields, respectively, and $\epsilon^{01} = 1$. The consistent anomaly satisfies the Wess-Zumino condition but the current J^μ transforms noncovariantly. We can define a new covariant current [13]

$$\tilde{J}^\mu = J^\mu \mp \frac{e^2}{4\pi\sqrt{-g}} A_\lambda \epsilon^{\lambda\mu} \tag{5}$$

which satisfies

$$\nabla_\mu \tilde{J}^\mu = \pm \frac{e^2}{4\pi\sqrt{-g}} \epsilon_{\mu\nu} F^{\mu\nu}. \tag{6}$$

The coefficient of the covariant anomaly is twice as large as that of the consistent anomaly.

The current is conserved $\partial_r J^r_{(o)} = 0$ outside the horizon. In the region near the horizon, since there are only outgoing (right-handed) fields, the current satisfies the anomalous equation

$$\partial_r J^r_{(H)} = \frac{e^2}{4\pi} \partial_r A_t. \tag{7}$$

Hence we can solve them in each region as

$$J^r_{(o)} = c_o, \tag{8}$$

$$J^r_{(H)} = c_H + \frac{e^2}{4\pi}[A_t(r) - A_t(r_+)], \tag{9}$$

where c_o and c_H are integration constants.

Under gauge transformations, variation of the effective action (without the omitted ingoing modes near the horizon) is given by $-\delta W = \int d^2x \sqrt{-g_{(2)}} \lambda \nabla_\mu J^\mu_{(2)}$, where λ is a gauge parameter. The current is written as a sum of two regions $J^\mu = J^\mu_{(o)} \Theta_+(r) + J^\mu_{(H)} H(r)$, where $\Theta_+(r) = \Theta(r - r_+ - \epsilon)$ and $H(r) = 1 - \Theta_+(r)$. Then, by using the anomaly equation, the variation becomes

$$-\delta W = \int d^2x \lambda \left[\delta(r - r_+ - \epsilon)\left(J^r_o - J^r_H + \frac{e^2}{4\pi} A_t \right) + \partial_r \left(\frac{e^2}{4\pi} A_t H \right) \right]. \tag{10}$$

The total effective action must be gauge invariant and the last term should be cancelled by quantum effects of the classically irrelevant ingoing modes. The quantum effect to cancel this term is the Wess-Zumino term induced by the ingoing modes near the horizon. The coefficient of the delta function should also vanish, which relates the coefficient of the current in two regions;

$$c_o = c_H - \frac{e^2}{4\pi} A_t(r_+). \tag{11}$$

c_H is the value of the consistent current at the horizon. In order to determine the current flow, we need to fix the value of the current at the horizon. Since the condition should be gauge covariant, we impose that the coefficient of the covariant current at the horizon should vanish. Since $\tilde{J}^r = J^r + \frac{e^2}{4\pi} A_t(r) H(r)$, that condition determines the value of the charge flux to be

$$c_o = -\frac{e^2}{2\pi} A_t(r_+) = \frac{e^2 Q}{2\pi r_+}. \tag{12}$$

This agrees with the current flow associated with the Hawking thermal (blackbody) radiation including chemical potential, as will appear presently.

Gravitational anomaly.—We now discuss the flow of the energy-momentum tensor. If we neglect quantum effects of the ingoing modes, the effective theory exhibits a gravitational anomaly. In $1+1$ dimensions the consistent anomaly reads [14,15]

$$\nabla_\mu T^\mu_\nu = \frac{1}{96\pi\sqrt{-g}} \epsilon^{\beta\delta} \partial_\delta \partial_\alpha \Gamma^\alpha_{\nu\beta} = \mathcal{A}_\nu, \tag{13}$$

for right-handed fields. The covariant anomaly, on the other hand, takes the form

$$\nabla_\mu \tilde{T}^\mu_\nu = \frac{1}{96\pi\sqrt{-g}}\epsilon_{\mu\nu}\partial^\mu R = \tilde{\mathcal{A}}_\nu. \tag{14}$$

Since the charged field ϕ is in a fixed background of the electric field and dilaton field σ, the energy-momentum tensor is not conserved even classically. We first derive the appropriate Ward identity. Under diffeomorphism transformations $x \to x' = x - \xi$, metric, gauge, and dilaton fields transform as $\delta g^{\mu\nu} = -(\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu)$, $\delta A_\mu = \xi^\nu \partial_\nu A_\mu + \partial_\mu \xi^\nu A_\nu$, and $\delta\sigma = \xi^\mu \partial_\mu \sigma$ and the action for matter fields $S[g_{\mu\nu}, A_\mu, \phi, \sigma]$ is invariant. Hence, if there were no gravitational anomaly, the partition function $Z = \int \mathcal{D}\phi \exp(iS)$ would obey

$$-i\int d^n x\bigg[\delta g^{\mu\nu}(x)\frac{\delta}{\delta g^{\mu\nu}(x)} + \delta A_\mu(x)\frac{\delta}{\delta A_\mu(x)} + \delta\sigma(x)\frac{\delta}{\delta\sigma(x)}\bigg]Z[g_{\mu\nu}, A_\mu, \sigma] = 0. \tag{15}$$

Using the energy-momentum tensor $T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}$ and current $J^\mu \equiv \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta A_\mu}$, the Ward identity becomes

$$\nabla_\mu T^\mu_\nu = F_{\mu\nu}J^\mu + A_\nu \nabla_\mu J^\mu - \frac{\partial_\nu\sigma}{\sqrt{-g}}\frac{\delta S}{\delta\sigma}. \tag{16}$$

Here we have used the fact that the dilaton couples to the Lagrangian itself, and kept the term proportional to the gauge anomaly. Adding the gravitational anomaly, the Ward identity becomes

$$\nabla_\mu T^\mu_\nu = F_{\mu\nu}J^\mu + A_\nu \nabla_\mu J^\mu - \frac{\partial_\nu\sigma}{\sqrt{-g}}\frac{\delta S}{\delta\sigma} + \mathcal{A}_\nu. \tag{17}$$

For a metric of the form (2), the anomaly is purely timelike ($\mathcal{A}_r = 0$) and $\mathcal{A}_t$ can be written as $\mathcal{A}_t \equiv \partial_r N^r_t$ where $N^r_t = (f'^2 + ff'')/192\pi$. The covariant anomaly is similarly written as $\tilde{\mathcal{A}}_t = \partial_r \tilde{N}^r_t$ where $\tilde{N}^r_t = (ff'' - (f')^2/2)/96\pi$. At the horizon, since $f = 0$, the coefficients have opposite signs.

We now solve the Ward identity (17) for the $\nu = t$ component. Since we are considering a static background, the contribution from the dilaton background can be dropped. In the exterior region without anomalies, the Ward identity is

$$\partial_r T^r_{t(o)} = F_{rt}J^r_{(o)}, \tag{18}$$

and by using $J^r_{(o)} = c_o$ it is solved as

$$T^r_{t(o)} = a_o + c_o A_t(r), \tag{19}$$

where a_o is an integration constant. Since there is a gauge and gravitational anomaly near the horizon, the Ward identity becomes

$$\partial_r T^r_{t(H)} = F_{rt}J^r_{(H)} + A_t \nabla_\mu J^\mu_{(H)} + \partial_r N^r_t. \tag{20}$$

The first and the second term can be combined to become $F_{rt}\tilde{J}^r_{(H)}$. By substituting $\tilde{J}^r_{(H)} = c_o + \frac{e^2}{2\pi}A_t(r)$ into this equation, $T^r_{t(H)}$ can be solved as

$$T^r_{t(H)} = a_H + \int_{r_+}^r dr\partial_r\bigg(c_o A_t + \frac{e^2}{4\pi}A_t^2 + N^r_t\bigg). \tag{21}$$

The energy-momentum tensor combines contributions from these two regions $T^\mu_\nu = T^\mu_{\nu(o)}\Theta_+ + T^\mu_{\nu(H)}H$. Under the diffeomorphism transformation, the effective action changes as

$$\int d^2x\sqrt{-g_{(2)}}\xi^t\nabla_\mu T^\mu_t = \int d^2x\xi^t\bigg[c_o\partial_r A_t(r) + \partial_r\bigg(\frac{e^2}{4\pi}A_t^2 + N^r_t\bigg) + \bigg(T^r_{t(o)} - T^r_{t(H)} + \frac{e^2}{4\pi}A_t^2 + N^r_t\bigg)\delta(r - r_+ - \epsilon)\bigg]. \tag{22}$$

The first term is the classical effect of the background electric field for constant current flow. The second term should be cancelled by the quantum effect of the incoming modes. The coefficient of the last term should vanish in order to restore the diffeomorphism covariance at the horizon. This relates the coefficients:

$$a_o = a_H + \frac{e^2}{4\pi}A_t^2(r_+) - N^r_t(r_+). \tag{23}$$

In order to determine a_o, we need to fix the value of the energy-momentum tensor at the horizon. As before, we impose a vanishing condition for the covariant energy-momentum tensor at the horizon. Since the covariant energy-momentum tensor is related to the consistent one by

$$\tilde{T}^r_t = T^r_t + \frac{1}{192\pi}[ff'' - 2(f')^2], \tag{24}$$

the condition reads $a_H = \kappa^2/24\pi = 2N^r_t(r_+)$, where $\kappa = 2\pi/\beta$ is the surface gravity of the black hole. The total flux of the energy-momentum tensor is given by

$$a_o = \frac{e^2Q^2}{4\pi r_+^2} + N^r_t(r_+) = \frac{e^2Q^2}{4\pi r_+^2} + \frac{\pi}{12\beta^2}. \tag{25}$$

Blackbody radiation.—Now we compare the results (12) and (25), with the fluxes from blackbody radiation moving in the positive r direction at an inverse temperature β with a chemical potential. The Planck distribution in the RN black hole is given by

$$I^{(\pm)}(w) = \frac{1}{e^{\beta(w\pm c)} - 1}, \qquad J^{(\pm)}(w) = \frac{1}{e^{\beta(w\pm c)} + 1}, \tag{26}$$

for bosons and fermions, respectively, and $c = eQ/r_+$ [1]. $I^{(-)}$ and $J^{(-)}$ correspond to the distributions for particles with charge e. In the zero temperature limit, if $(\omega \pm c)$ is positive, those distributions are suppressed exponentially. But if it is negative, they become ∓ 1 for bosons or fermions. This result needs further interpretation, especially for bosons. In the bosonic case the absorption coefficient also becomes negative, leading to the effect known as superradiance [16]. In the more straightforward fermionic case, occupation numbers for these low frequency modes become 1 at zero temperature, which leads to a nonzero flux of radiation even at the extremal case.

To keep things simple, we focus now on the fermion case. With these distributions, the flux of current and energy-momentum become

$$J^r = e\int_0^\infty \frac{dw}{2\pi}[J^-(w) - J^{(+)}(w)] = \frac{e^2 Q}{2\pi r_+}, \qquad (27)$$

$$T^r_t = \int_0^\infty \frac{dw}{2\pi} w[J^-(w) + J^{(+)}(w)] = \frac{e^2Q^2}{4\pi r_+^2} + \frac{\pi}{12\beta^2}. \qquad (28)$$

The results (12) and (25), derived from the anomaly cancellation conditions coincide with these results (27) and (28). Thus, the thermal flux required by black hole thermodynamics is capable of cancelling the anomaly. The actual emission is obtained by propagating the emission from these sources through the effective potential due to spatial curvature outside the horizon. The resulting radiation observed at infinity is that of a d-dimensional gray body at the Hawking temperature.

Discussion.—We have derived the flow of charge and energy momentum from charged black hole horizons. In contrast to the conformal anomaly derivation [17], we did not need to determine the current or energy-momentum tensor elsewhere. This is consistent with the universality of Hawking radiation. Solving the $\nu = r$ component of the Ward identity (17) would require detailed information on the microscopic Lagrangian; likewise other components like T^r_r are strongly dependent on such nonuniversal physics.

We have assumed that covariant forms of current or energy-momentum tensor should vanish at the horizon. This is natural since physical conditions should be gauge or diffeomorphism invariant. But we would like to understand more deeply why we should use the covariant forms instead of the consistent ones for boundary conditions at the horizon. In [5], which employed different procedures (integrating out modes in a sandwich surrounding the horizon), the consistent anomaly appeared. The same results (12) and (25), can be also derived by calculating the effective action for two-dimensional free fields in the RN black hole background and imposing regularity at the horizon [18]. This indicates that the regularity is closely related to the vanishing conditions of covariant currents.

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