Time Dependence of Correlation Functions Following a Quantum Quench

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We show that the time dependence of correlation functions in an extended quantum system in d dimensions, which is prepared in the ground state of some Hamiltonian and then evolves without dissipation according to some other Hamiltonian, may be extracted using methods of boundary critical phenomena in $d+1$ dimensions. For $d=1$ particularly powerful results are available using conformal field theory. These are checked against those available from solvable models. They may be explained in terms of a picture, valid more generally, whereby quasiparticles, entangled over regions of the order of the correlation length in the initial state, then propagate classically through the system.

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Suppose that an extended quantum system in d dimensions (for example, a quantum spin system) is prepared at time $t=0$ in a pure state $|\psi_0\rangle$ which is the ground state of some Hamiltonian H_0 (or, more generally, in a thermal state at a temperature less than the gap m_0 to the first excited state.) For times $t>0$ the system evolves *unitarily* according to the dynamics given by a *different* Hamiltonian H, which may be related to H_0 by varying a parameter such as an external field. This variation, or quench, is supposed to be carried out over a time scale much less than m_0^{-1}. How do the correlation functions, expectation values of products of local observables, then evolve? The answer to this question would appear to depend in detail on the system under consideration. It was first addressed in the context of the quantum Ising-XY model in Refs. [1] [see also [2]]. Until recently it was, however, largely an academic question, because the time scales over which most condensed matter systems can evolve coherently without coupling to the local environment are far too short, and the effects of dissipation and noise are inescapable. However, with the development of experimental tools for studying the behavior of optical lattices of ultracold atoms, and quantum phase transitions in these systems [3], there has been renewed interest in this theoretical problem [see, for example, [4,5]].

In this Letter we study such problems in general and argue that, if H is at or close to a quantum critical point (while H_0 is not), there is a large degree of universality in the behavior at sufficiently large distances and late times, despite the fact that correlations typically fall off exponentially, rather than the power laws characteristic of the ground state near a quantum critical point. Our arguments are based on the path integral approach and the well-known mapping of the quantum problem to a classical one in $d+1$ dimensions. The initial state plays the role of a boundary condition, and we are able to then use the renormalization group (RG) theory of boundary critical behavior [see, e.g., [6]]. From this point of view, particularly powerful analytic results are available for $d=1$ and when the quantum critical point has dynamic exponent $z=1$ (or, equivalently, a linear quasiparticle dispersion relation $\omega = v|k|$) because then the $1+1$-dimensional problem is described asymptotically by a boundary conformal field theory (BCFT) [7,8]. Some of these methods have recently been applied [9] to studying the time evolution of the entanglement entropy, but, as we shall argue, they are more generally applicable. Further details of these calculations will appear elsewhere [10].

The results we find from CFT suggest a rather simple picture which is, however, more generally applicable: the state $|\psi_0\rangle$, which has an (extensively) high energy compared with that of the ground state of H, acts as a source for quasiparticle excitations. Those quasiparticles originating from closely separated points (roughly within the correlation length ξ_0 of the ground state of H_0) are quantum entangled. However, once they are emitted they behave semiclassically, traveling at speed v. They have two distinct effects. First, incoherent quasiparticles arriving a given point $\mathbf{r}$ from well-separated sources cause relaxation of (most) local observables at $\mathbf{r}$ towards their ground-state expectation values. (An exception is the local energy density which, of course, is conserved.) In the CFT case this relaxation is exponential $\sim \exp(-\pi x v t/2\tau_0)$. Here $\tau_0 \sim m_0^{-1}$ is nonuniversal, but x, the bulk *scaling dimension* of the particular observable, is related to the critical exponents of the quantum phase transition, and is universal. Hence *ratios* of decay constants for different observables should be universal. Second, entangled quasiparticles arriving at the same time t at points with separation $|\mathbf{r}| \gg \xi_0$ induce (quantum) correlations between local observables. In the case where they travel at a unique speed v, therefore, there is a sharp ''light-cone'' effect: the connected correlations do not change significantly from their initial values until time $t \sim |\mathbf{r}|/2v$. In the CFT case this light-cone effect is rounded off in a (calculable) manner over the region $t - |\mathbf{r}|/2v \sim \tau_0$, since quasiparticles remain entangled over this distance scale. After this they rapidly saturate to *time-independent* values. For large separations (but still

$\ll 2vt$), these decay *exponentially* $\sim \exp(-\pi x|\mathbf{r}|/2v\tau_0)$. Thus, while the generic one-point functions relax to their ground-state values, the correlation functions do not, because, at quantum criticality, these would have a power law dependence. Of course, this is to be expected since the mean energy is much higher than that of the ground state, and it does not relax.

These results rely on the technical assumption that the leading asymptotic behavior given by CFT, which applies to the Euclidean region (large imaginary times), may simply be analytically continued to find the behavior at large real time. While such procedures have been shown to give the correct behavior for the time dependent correlations in equilibrium, it is important to check them in specific solvable cases. We have done this for the case of a lattice free boson (coupled harmonic oscillators), and for the Ising-XY chain, which can be transformed into a free fermion problem. These models, which are exactly solvable on the lattice, confirm our general results and also show how the semiclassical picture discussed above is modified for a more general quasiparticle dispersion relation $\omega = \Omega_k$, taking into account both the effects of the lattice and of a finite gap. The results are consistent with this picture as long as the quasiparticles are assumed to propagate at the group velocity Ω'_k appropriate to their wave number k. In this case the light-cone effect first occurs at time $t \sim |\mathbf{r}|/2v_m$, where v_m is the maximum group velocity. (If this occurs at a nonzero wave number, it gives rise to spatial oscillations in the correlation function.) However, because there are also quasiparticles moving at speeds less than v_m, the approach to the asymptotic behavior at late times is less abrupt. In fact, for a lattice dispersion relation where Ω'_k vanishes at the zone boundary, the approach to the limit is slow, as an inverse power of t. A similar result applies to the 1-point functions. This is consistent with the exact results found in [1].

We first discuss some general features of the path integral approach and its relation to boundary critical behavior. The expectation value of any product of local operators is given by

$$\langle \mathcal{O}(t, \{\mathbf{r}_i\})\rangle = Z^{-1}\langle\psi_0|e^{iHt-\epsilon H}\mathcal{O}(\{\mathbf{r}_i\})e^{-iHt-\epsilon H}|\psi_0\rangle, \quad (1)$$

where we have included damping factors $e^{-\epsilon H}$ in such a way as to make the path integral representation of the expectation value absolutely convergent. The normalization factor is $Z = \langle\psi_0|e^{-2\epsilon H}|\psi_0\rangle$. Equation (1) may be represented by a path integral in imaginary time τ

$$\frac{1}{Z}\int[d\phi(\mathbf{r},\tau)]\mathcal{O}(\{\mathbf{r}_i\},0)e^{-S[\phi]}\langle\psi_0|\phi(\mathbf{r},\tau_2)\rangle\langle\phi(\mathbf{r},\tau_1)|\psi_0\rangle \quad (2)$$

over a complete set of fields $\phi(\mathbf{r},\tau)$ (or, in a spin system, a coherent state representation), with $S = \int_{\tau_1}^{\tau_2} L d\tau$, analytically continued to $\tau_1 = -\epsilon - it$ and $\tau_2 = \epsilon - it$. Here L is the (Euclidean) Lagrangian corresponding to the dynamics of H. We will consider the equivalent slab geometry between $\tau = 0$ and $\tau = 2\epsilon$, with $\mathcal{O}$ inserted at $\tau = \epsilon + it$.

Equation (2) has the form of the equilibrium expectation value in a $d+1$-dimensional slab geometry with particular boundary conditions. We wish to study this in the limit when t and the separations $|\mathbf{r}_i - \mathbf{r}_j|$ are much larger than the microscopic length and time scales, when RG theory can be applied. If H is at or close to a quantum critical point, the bulk properties of the critical theory are described by a bulk RG fixed point (or some relevant perturbation thereof). In that case, the boundary conditions flow to one of a number of possible *boundary* fixed points [6]. Thus, for the purpose of extracting the asymptotic behavior, we may replace $|\psi_0\rangle$ by the appropriate RG-invariant boundary state $|\psi_0^*\rangle$ to which it flows. The difference may be taken into account, to leading order, by assuming that the RG-invariant boundary conditions are not imposed at $\tau = 0$ and $\tau = 2\epsilon$ but at $\tau = -\tau_0$ and $\tau = 2\epsilon + \tau_0$. In the language of boundary critical behavior, τ_0 is called the extrapolation length [6]. It characterizes the RG distance of the actual boundary state from the RG-invariant one. It is always necessary because scale-invariant boundary states are not in fact normalizable [8]. It is expected to be of the order of the typical time scale of the dynamics near the ground state of H_0, that is the inverse gap m_0^{-1}. The effect of introducing τ_0 is simply to replace ϵ by $\epsilon + \tau_0$. The limit $\epsilon \to 0+$ can now safely be taken, so the width of the slab is then taken to be $2\tau_0$.

$d = 1$ and CFT.—The above was completely general, but we now consider the case when H is at a quantum critical point whose long-distance behavior is given by a $1+1$-dimensional CFT, with dispersion relation $\omega = v|k|$. We set $v = 1$ in the following. RG-invariant boundary conditions then correspond to conformally invariant boundary states. Under these circumstances the correlation functions of local operators in the slab geometry (whose points are labeled by a complex number w with $0 < \mathrm{Im}\, w < 2\tau_0$) are related to those in a half-space $\mathrm{Im}\, z > 0$ by the conformal mapping $w = (2\tau_0/\pi)\log z$. In the case where $\mathcal{O}$ is a product of local *primary* scalar operators $\Phi_i(w_i)$ (which covers most of the physically relevant cases, for important exceptions see below) we have

$$\left\langle\prod_i \Phi_i(w_i)\right\rangle_{\text{strip}} = \prod_i |w'(z_i)|^{-x_i}\left\langle\prod_i \Phi_i(z_i)\right\rangle_{\text{UHP}}, \quad (3)$$

where x_i is the bulk scaling dimension of Φ_i. Note that the expectation values of the Φ_i in the ground state of H are supposed to have been subtracted off. We now discuss some special cases of (3).

One-point functions.—In this case scale invariance implies that $\langle\Phi(z)\rangle_{\text{UHP}} = A_b^\Phi(2\,\mathrm{Im}\, z)^{-x}$, where A_b^Φ is an amplitude depending on the selected boundary condition. (It may vanish if Φ corresponds to an operator whose expectation value in $|\psi_0\rangle$ vanishes.) This gives

$$\langle\Phi(w)\rangle_{\text{strip}} = A_b^\Phi\left[\frac{\pi}{4\tau_0}\frac{1}{\sin[\pi\tau/(2\tau_0)]}\right]^x. \quad (4)$$

Continuing to $\tau = \tau_0 + it$ we then find

$$\langle \Phi(t)\rangle = A_b^\Phi\left[\frac{\pi}{4\tau_0}\frac{1}{\cosh[\pi t/(2\tau_0)]}\right]^x, \qquad (5)$$

which exhibits exponential decay for $t \gg \tau_0$, with a decay time related to the critical exponent x, as described in the introduction.

An important exception to this law is the local energy density (or any piece thereof). This corresponds to the tt component of the energy-momentum tensor $T_{\mu\nu}$. In CFT this is not a primary operator. Indeed, if it is normalized so that $\langle T_{\mu\nu}\rangle_{\rm UHP} = 0$, in the strip [11] $\langle T_{tt}(\mathbf{r}, \tau)\rangle = \pi c/24(2\tau_0)^2$ (where c is the central charge of the CFT) so that it does not decay in time. Of course, this is to be expected since the dynamics conserves energy. A similar feature is expected to hold for other local densities corresponding to globally conserved quantities which commute with H, for example, the total spin.

Two-point functions.—The 2-point function in the half plane takes the general form [7]

$$\langle \Phi(z_1)\Phi(z_2)\rangle_{\rm UHP} = \left(\frac{z_{1\bar 2}z_{2\bar 1}}{z_{12}z_{\bar 1\bar 2}z_{1\bar 1}z_{2\bar 2}}\right)^x F(\eta), \qquad (6)$$

where $z_{ij} = z_i - z_j$, $z_{\bar i}$ is the image of z_i in the real axis, and $\eta \equiv z_{1\bar 1}z_{2\bar 2}/z_{1\bar 2}z_{2\bar 1}$. The function F is universal but depends in detail on the particular BCFT. Using (3), analytically continuing, and assuming $r, t \gg \tau_0$ we find

$$\langle \Phi(0,t)\Phi(r,t)\rangle = (\pi/2\tau_0)^{2x}\left(\frac{e^{\pi r/2\tau_0} + e^{\pi t/\tau_0}}{e^{\pi r/2\tau_0}e^{\pi t/\tau_0}}\right)^x F(\eta), \qquad (7)$$

where now

$$\eta \sim \frac{e^{\pi t/\tau_0}}{e^{\pi r/2\tau_0} + e^{\pi t/\tau_0}}. \qquad (8)$$

Thus for $r - 2t \gg \tau_0$, $\eta \sim e^{\pi(t-r/2)/\tau_0} \ll 1$, while for $2t - r \gg \tau_0$, $\eta \sim 1$. In both of these limits the behavior of F is determined by the short-distance expansion. The first limit, when $t < r/2$, corresponds in the half plane to both points being close to the boundary, when we can use the bulk-boundary expansion to argue that $F(\eta) \sim (A_b^\Phi)^2\eta^{x_b}$, where x_b is the boundary scaling dimension of the leading boundary operator to which Φ couples. This gives, for $t < r/2$,

$$\langle \Phi(r,t)\Phi(0,t)\rangle \sim (A_b^\Phi)^2 e^{-\pi x t/\tau_0}e^{-\pi x_b(r/2-t)/\tau_0}. \qquad (9)$$

Note that if $\langle\Phi\rangle \neq 0$, $x_b = 0$ and (9) is just $\langle\Phi\rangle^2$. In that case the leading behavior of the *connected* 2-point function is given by subleading terms either in F or in the bulk-boundary short-distance expansion. The opposite limit, when $t > r/2$, corresponds to the two points both being far from the boundary, in which case the bulk behavior dominates and $F \to 1$. This gives

$$\langle \Phi(r,t)\Phi(0,t)\rangle \sim e^{-\pi x r/2\tau_0}, \qquad (10)$$

that is, the correlations saturate (exponentially fast) for $t > r/2$. Note, however, that the precise details of the crossover behavior for $|t - r/2| \sim \tau_0$ are dependent on the form of F for $0 < \eta < 1$.

Other CFT results.—Many other general results may be found within the CFT formalism. For example, the 2-time correlation function

$$\langle \Phi(r,t)\Phi(0,s)\rangle \sim \begin{cases} e^{-\pi x(t+s)/4\tau_0} & \text{for } r > t+s, \\ e^{-\pi x r/4\tau_0} & \text{for } t-s < r < t+s, \\ e^{-\pi|t-s|/4\tau_0} & \text{for } r < |t-s|, \end{cases} \qquad (11)$$

where it is assumed that t, s, $|t-s|$, and r are all $\gg \tau_0$. In the case where $\langle\Phi\rangle = 0$, the first case gains an additional factor $e^{-\pi x_b(t+s-r)/4\tau_0}$. Note that the autocorrelation function with $r = 0$ depends only on the time difference and does not exhibit aging in this regime. Another example is the 1-point function in a semi-infinite chain, for which we find

$$\langle \Phi(r,t)\rangle \sim \begin{cases} e^{-\pi x t/2\tau_0} & \text{for } t < r, \\ e^{-\pi x r/2\tau_0} & \text{for } t > r. \end{cases} \qquad (12)$$

The Gaussian chain.—The simplest exactly solvable model in which to study these effects is a general chain of coupled harmonic oscillators with a Hamiltonian

$$\frac{1}{2}\sum_r\left[\pi_r^2 + m^2\phi_r^2 + \sum_j \omega_j^2(\phi_{r+j} - \phi_r)^2\right] \qquad (13)$$

with the standard commutation relations imposed between the ϕ_r and their canonical conjugates π_r. The 2-point function can be found straightforwardly if tediously by integrating the Heisenberg equations of motion for the Fourier modes. The result is

$$\langle \phi_r(t)\phi_0(t)\rangle - \langle \phi_r(0)\phi_0(0)\rangle = \int_{\rm BZ} e^{ikr}\frac{(\Omega_{0k}^2 - \Omega_k^2)[1-\cos(2\Omega_k t)]}{\Omega_k^2\Omega_{0k}}dk, \qquad (14)$$

where Ω_{0k} and Ω_k are the single-particle dispersion relations corresponding to H_0 and H, respectively, and the integral is over the first zone. The CFT results correspond to the case with $\Omega_k \sim v|k|$ as $k \to 0$, which gives $\langle \phi_r(t)\phi_0(t)\rangle \sim m_0(t - r/2v)$ for $m_0(t - r/2v) \gg 1$, while it vanishes for $t < r/2v$. Taking Φ to be the conformal field $e^{iq\phi}$ then reproduces all the CFT results before, with $\tau_0 \sim m_0^{-1}$, $x \propto q^2$, and $F = 1$.

However, the advantage of this simple model is that the effects of other dispersion relations can be understood. In fact, the dominant contribution to (14) in the limits of large t and r comes from where, by a stationary phase argument, the group velocity $v_k \equiv \Omega_k' = r/2t$. This suggests the picture given in the introduction of the correlations being due to left- and right-moving particles emitted at $t = 0$ from closely separated points, moving with the group velocity. It also explains for the case of a semi-infinite chain why the relevant time scale is r/v rather than $r/2v$, since one of the particles is reflected from the end of the chain. For the energy density, proportional to $(\phi_{r+1} - \phi_r)^2$, the asymptotic behavior is a constant, but the approach to this is dominated by the particles with the smallest group velocity, which, for a gapless lattice model, come from the zone boundary at $|k| = \pi$. An explicit calculation [10] gives a correction $\sim t^{-3/2}\cos(\Omega_\pi t + \pi/4)$. Similarly, for a quench to a *gapped* lattice H, with $\Omega_0 > 0$, the fastest particles correspond to a nonzero wave number. This gives rise to spatial oscillations in the correlation function.

The Ising- XY chain.—As an example, consider the CFT describing the scaling limit of the Ising-XY chain with Hamiltonian

$$\sum_{r=1}^{N}\left(\frac{(1+\gamma)}{2}\sigma_r^x\sigma_{r+1}^x + \frac{(1-\gamma)}{2}\sigma_r^y\sigma_{r+1}^y - h\sigma_r^z\right), \quad (15)$$

where γ is the anisotropy parameter and h is the external transverse field. It is well known that for any $\gamma \neq 0$ the model undergoes a phase transition at $h = 1$ that is in the universality class of the Ising model (defined by $\gamma = 1$). In that case, F is known exactly [7] to be $F = (\sqrt{1+\eta^{1/2}} \pm \sqrt{1-\eta^{1/2}})/\sqrt{2}$, where the upper and lower signs correspond, respectively, to fixed and free boundary conditions on the spins σ_r^x. These correspond to quenches to the critical point $h = 1$ from $h_0 < 1$ and $h_0 > 1$, respectively. Our general results should then apply, for example, to the correlations of the order parameter σ^x, with scaling dimensions $x = \frac{1}{8}$ and $x_b = \frac{1}{2}$ for $h_0 > 1$ and zero for $h_0 < 1$.

The time dependence of the transverse magnetization $\langle\sigma_r^z(t)\rangle$ (which is, in fact, a piece of the energy density) has been studied in Ref. [1]. For case of a quench from $h_0 > 1$ to $h = 1$ the results agree precisely with our general results of a constant asymptotic value approached by an oscillatory $t^{-3/2}$ power law, which we have argued above is due to lattice effects. In Ref. [4] the time averaged behavior of the spin-spin correlation function was calculated for quenches from $h_0 = 0$ and infinity. For quenches to the critical theory with $h = 1$, the behavior was found to be purely exponential, in agreement with (10). In Ref. [2] the behavior of the correlation function in a finite system was studied numerically. While in this case there are complications from both the finite size and lattice effects, the main features are once again consistent with our general theory.

We conclude with a discussion of the expected behavior in higher dimensions $d > 1$. For quenches from one disordered state to another outside the critical region the Gaussian model results (14) (with $\int dk \to \int d^dk$) should apply. For $t < r/2v_m$ the 2-point function behaves as at $t = 0$. At large times it decays exponentially with a scale set by the correlation length of H (although its Fourier transform differs in detail from the behavior in the ground state of H). The approach to this for $t > r/2v_m$ is, however, more complicated, since the wave fronts are no longer planar. A similar analysis may be applied to the dynamics of the Goldstone modes, or spin waves, in the quench to an ordered state. Above the upper critical dimension ($d = 3$ if $z = 1$) mean-field theory should also apply to quenches to the critical point. In $d = 3$, this gives $\langle\phi(\mathbf{r}, t)\phi(0, t)\rangle \sim r^{-1}e^{-\pi(t-r/2v)/\tau_0}$ for $t < r/2v$, and saturates at later times to t-independent value. Similarly we find, analytically continuing the results in Ref. [12] for a slab geometry, that for a critical quench from a disordered state, the energy density parameter $\langle\phi^2\rangle$ decays exponentially. In principle it should also be possible to use the results in [13] for a slab geometry in $d = 3 - \epsilon$ dimensions, but this is very cumbersome. The calculation is simpler at large N [12] and once again shows exponential decay. However, a counterexample to this general rule appears to be given by the behavior of the magnetization in a critical quench from an ordered state. The mean-field profile in a slab geometry, given in [14], involves elliptic functions, which, when continued to real time, give nondecaying oscillations with period $\sim\tau_0$. We expect this result to be modified by the inclusion of fluctuations.

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