

Statistics of Crumpled Paper

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A statistical study of crumpled paper is allowed by a minimal 1D model: a self-avoiding line bent at sharp angles—in which the elastic energy resides—put in a confining potential. Many independent equilibrium configurations are generated numerically and their properties are investigated. At small confinement, the distribution of segment lengths is log-normal in agreement with previous predictions and experiments. At high confinement, the system approaches a jammed state with a critical behavior, whereas the length distribution follows a gamma law in which the parameter is predicted as a function of the number of layers in the system.

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When a sheet of paper is crushed, the formation of a network of ridges is observed. The process is irreversible as energy accumulates in small regions leading to localized plastic flow. Many fundamental questions arise. What is the resistance of the crumpled ball to mechanical forces? What are the mechanisms of the cascade of energy to small scales? What is the probability distribution of lengths and energies? The two latter are similar to the questions central to hydrodynamic turbulence [1]. Early experimental studies [2–4] of crumpled paper balls focused on fractal properties such as the scaling of the radius with the size of the flat sheet. The same property was used to characterize the phases of microscopic membranes [5]—such as red blood cells or graphitic oxide—for which thermal fluctuations are relevant; these microscopic membranes raise a number of numerical and theoretical difficulties, particularly when self-avoidance is implemented [6]. Crumpled paper was also viewed as a self-affine surface whose roughness was predicted and measured [7,8].

From elasticity theory [9], we know that thin elastic plates have two modes of deformation: bending, which involves curving the plate, and stretching, which changes the distances on the plate. Bending is much less expensive energetically than stretching so that pure bending deformations should be always preferred. However, this is not possible for a number of boundary conditions as shown in [10]; this leads to the formation of a near-singular network of lines (stretching ridges) and points (developable cones) where the expensive stretching energy is localized. Even if isolated ridges [11–13] or d cones [10,14–16] are rather well characterized, the understanding of a full network is far from being complete. Most experimental and theoretical studies tackled situations with a small number of singularities [14,17–20] or a highly symmetric network [21,22].

Our aim is to investigate numerically and theoretically the statistics of crumpled paper. The strength of crumpled sheets was measured and found to involve logarithmic relaxation and a critical behavior close to a compact con-

figuration [23]. Experiments also showed a broad distribution for the length of ridges, either indirectly—through noise emission [24,25]—or directly—through geometrical measurements [26]. This could be explained [27] in terms of the hierarchical random breaking of ridges into smaller ones. However, full numerical approaches [12,28–30] are impeded by the complexity of the problem and did not allow any statistics. Here we introduce a minimal 1D model. We generate numerically a large number of equilibrium configurations and study the resulting mechanical and geometrical properties. In particular, we show a transition of the length distribution from log-normal to gamma and we predict the parameter of the gamma distribution using arguments analogous to those of mixing [31] and spray formation [32].

A 2D crumpled sheet can be considered as a self-avoiding polyhedron such that its energy is concentrated at the edges (stretching ridges [13]) and vertices (d cones [10]). The 1D equivalent that we introduce is a self-avoiding closed line bent at sharp angles and in which energy is concentrated at the vertices [Fig. 1(a)]. The important feature here is that the position of the vertices along the line is free. This model can be viewed either as a representation of a cut through a crumpled ball of paper [Fig. 1(b)] or as accounting for a crumpled sheet such that all ridges are parallel and have the same length L . The state of the system is defined by the coordinates (x'_i, y'_i) of the N vertices. The total length of the line is conserved and set to 1 by defining its energy as a function of the rescaled coordinates $(x_i, y_i) = (x'_i, y'_i)/l$, $l = \sum_i [(x'_{i+1} - x'_i)^2 + (y'_{i+1} - y'_i)^2]^{1/2}$. In order to keep the features of 2D sheets, the energy of each vertex is taken as the energy of a ridge (an edge) bent with the same angle ϕ_i —defined so that $\phi_i = 0$ when the ridge is flat—and of length L (the value of L is unimportant as it will be scaled out). Note that the exact form of the ridge energy is unimportant—see discussion at the end. The line is put in a confining quadratic potential of strength λ (this form is better for the convergence of the minimization but it does not affect the results,

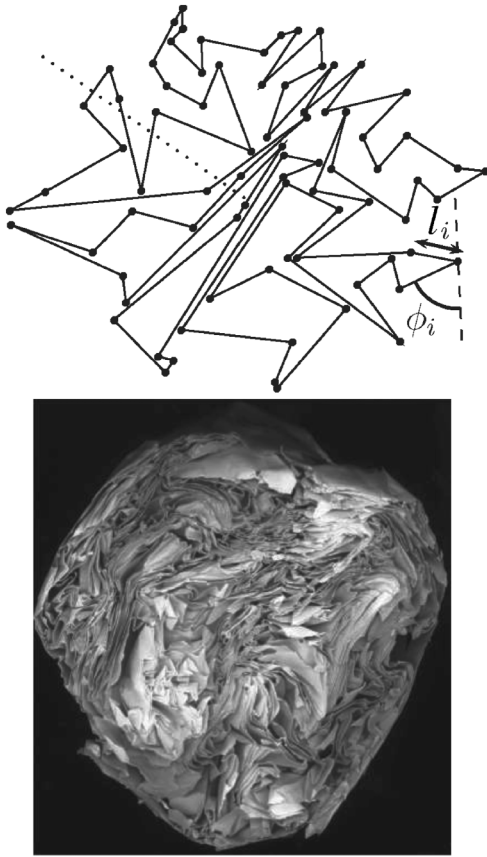


FIG. 1. (a) A compact equilibrium state of the 1D model. A self-avoiding closed line bent at sharp angles ϕ_i with N segments of free lengths l_i is put in a confining quadratic potential of strength λ . Here the number of vertices is $N = 80$, the thickness $h = 10^{-3}$ (in units of total line length) and the confinement $\lambda = 10^6$. (b) Picture of a cut through a crumpled ball of paper, courtesy E. Couturier.

see discussion at the end). The physical constraint of self-avoidance is implemented through a hard-core interaction \mathcal{E}_{hc} with infinite cost (a large number in the simulation) between all pairs of segments, taking into account their width h . The total energy reads

$$\mathcal{E} = \sum_i \tan^{7/3} \phi_i + \lambda \oint \mathbf{r}^2(s) ds + \mathcal{E}_{\text{hc}}. \quad (1)$$

$\mathbf{r}(s)$ is the (rescaled) position vector as a function of the curvilinear coordinate along the line. The unit of energy is the bending modulus of the sheet multiplied by the factor $(L/h)^{1/3}$ that comes from the ridge energy [13]; the unit of length is the total length of the line. The number of vertices N and the thickness h —which appears only in the hard-core interaction—are left as parameters whereas the confinement λ is the control parameter. This system has a large number of energy minima—which is favored by self-avoidance—and our aim is to investigate the statistical properties of the corresponding equilibrium configurations.

In the numerical minimization of the energy, Powell's algorithm [33] appeared as the most convenient to cope

with the discontinuities of the hard-core interaction. This algorithm is built upon a set of M directions (M being the number of degrees of freedom) which are used to determine the directions for the successive 1D minimizations. In order to obtain independent samples of the metastable states of the system, the set of directions is chosen at random. A better convergence was obtained by running Powell's algorithm 4 times and initializing the set of directions before each run. Care was taken in optimizing the evaluation of the hard-core interaction which is expensive (typically 90% of CPU time) because it involves all pairs of segments. The results reported here correspond to 4 values of $N \times 6$ values of the thickness h (in the range 10^{-5} – 10^{-2}) $\times 150$ samples $\times 40$ values of the confinement $\sim 10^5$ minimizations with 100–200 degrees of freedom.

We first measured the gyration radius $R_g = \langle r \rangle$ averaged over all samples as a function of the confinement λ (e.g., Fig. 2 for $N = 80$). For a given sample, the gyration radius depends on the choice of the sequence of applied confinements $\lambda = \lambda_1, \lambda_2 \dots$ as in the experiments of [23]. This can be ascribed to the large number of metastable states so that the equilibrium configuration is selected by the history of the system. The process of averaging over samples almost suppresses this dependence. At low lambda, the equilibrium state is a regular closed polyhedron so $2\pi R_g \approx 1$. When λ reaches a threshold $\lambda_c \sim 1$, the polyhedron buckles and starts to fold. When $\lambda \sim 100$, many contact points have formed. Eventually R_g decreases slowly with λ to a minimal value R_g^j value such that the system is jammed; we roughly find $(R_g - R_g^j) \sim \lambda^{-\alpha}$, with $\alpha = 0.3 \pm 0.1$. In this large λ regime, we also found the total energy \mathcal{E} and the energy of the ridges \mathcal{E}_r to scale as $\mathcal{E} \sim \lambda^\beta$, $\mathcal{E}_r \sim \lambda^{\beta_r}$, with $\beta = 1 \pm 0.05$ and $\beta_r = 0.45 \pm 0.05$. The value of α differs from the experimental [23] and numerical [30] value $\alpha = 0.54$ for 2D sheets probably because of the different dimensionality. The value of $\beta \approx 1$ shows that the system is almost jammed as most of its energy comes from the confining potential.

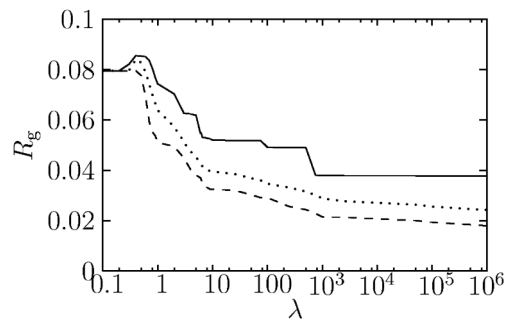


FIG. 2. Gyration radius of the line R_g as a function of the strength of the confining potential λ . Dotted (dashed) line: average over 60 samples; continuous line: maximum (minimum) value of R_g among all samples. $N = 80$ and $h = 10^{-3}$. The differences between maximum and minimum are relatively large; nevertheless, the standard deviation is only of the order of 510^{-4} .

We examined the correlations of the normals along the line and we found them to be small except at high confinement or for zero thickness as some pairs of segments in contact are exactly aligned. The probability distribution of angles is symmetric with respect to zero and almost flat—of course this probability vanishes for the maximal angles $\phi = \pm\pi/2$. In contrast, the distribution of distances l between two consecutive vertices appeared to be broad. We first checked that a threshold on length (up to $10h$) or angle (up to 10°)—i.e., suppressing from the statistics small segments or vertices with a small angle—does not affect the results.

At small confinement ($1 < \lambda \leq 10^4$ or $1 > 2\pi R_g \geq 0.2$), we observe sequential buckling events such that a part of the line folds inward reducing the radius of the ball. As predicted in [27] and found experimentally in [26], the hierarchical splitting of a line leads to a log-normal distribution [see Fig. 3(a)] with probability density

$$P_{\text{LN}}(x = l/\langle l \rangle) = \frac{1}{\sigma x \sqrt{\pi}} \exp\left[-\left(\frac{\ln x}{\sigma} + \frac{\sigma}{4}\right)^2\right]. \quad (2)$$

This distribution is in agreement with our data (except for sizes comparable to the thickness h), with a width σ in the range 1.0–1.8, comparable to the experimental values in the range 1.2–1.4 [26]. Note that these experiments were

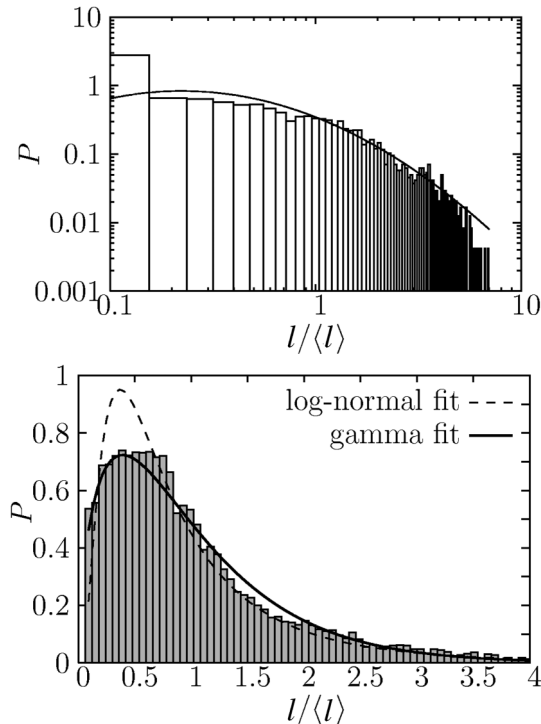


FIG. 3. The distribution of segments lengths $P(l/\langle l \rangle)$ normalized by the mean length ($N = 60$ and $h = 10^{-3}$). (a) Low confinement $\lambda = 900$. The line is a best fit to a log-normal distribution P_{LN} [Eq. (2)] with width $\sigma = 1.59 \pm 0.03$. (b) High confinement $\lambda = 10^6$. The two lines are a best fit to a log-normal distribution P_{LN} [Eq. (2)] and to a gamma distribution P_{Γ} [Eq. (3)] with parameter $\alpha = 1.64 \pm 0.03$.

performed at low confinement, $2\pi R_g \geq 0.6$ in units of sheet length.

In contrast, at high confinement ($\lambda \geq 10^4$ or $2\pi R_g \leq 0.2$), we found that a gamma distribution with density

$$P_{\Gamma}(x = l/\langle l \rangle) = \frac{(\alpha x)^{\alpha}}{\Gamma(\alpha)x} \exp(-\alpha x) \quad (3)$$

provides a better fit to the data than a log-normal distribution which fails for the most likely sizes [Fig. 3(b)] and for the large l tail of the distribution which is exponential; the log-normal distribution is however acceptable at intermediate sizes [Fig. 3(b)]. This can be accounted for as follows. The buckling process is no longer hierarchical which at first sight should lead to an exponential distribution of lengths $P_{\text{E}}(x = l/\langle l \rangle) = \exp(-x)$ as the result of the random splitting of a line [34]. However, at large confinement, many self-contact points exist and segments in contact tend to correlate. Let us define the number of layers N_{L} as the mean number of intersections of the system with a straight half line starting from the center of the potential (see Fig. 1), averaged over all directions and all samples. For our parameters, N_{L} varies from 1 at low confinement to about 5 at the largest confinements (except in the case $h = 0$: N_{L} can be as large as 7.5). Note that the number of layers is always smaller than the simple estimate $1/(2\pi R_g)$ (in the range 1–8) which reflects the “roughness” of the ball. The system can be viewed as N_{L} interacting layers. If they were independent, the distribution of lengths would be exponential P_{E} . Here the lengths of segments in contact tend to average. So the length should be the mean of N_{L} exponential random variables, i.e., a random variable with a Gamma distribution P_{Γ} of parameter $\alpha = N_{\text{L}}$. We measured the parameter from the fit to the numerical distributions and we found that, as soon as $N_{\text{L}} \geq 2.5$,

$$\alpha = aN_{\text{L}} - b \quad (4)$$

with $a = 0.95 \pm 0.1$ and $b = 2.05 \pm 0.2$. This is in agreement with our argument (Fig. 4) except that the effective number of layer is smaller by 2 than N_{L} , which simply means that the system is slightly less correlated than if two segments in contact had exactly the same length.

To summarize, we have introduced a simple 1D model which allows a comprehensive statistical study of crumpled paper. At low confinement, a hierarchy of buckling events leads to a log-normal distribution of lengths in agreement with the predictions of [27] and the experiments of [26]. At high confinement, jamming is approached with a critical behavior; self-avoidance leads to self-correlations and to a gamma distribution of lengths solely determined by the number of layers in the system (Fig. 4). This increase in correlations probably accounts for the Hurst exponent of compact paper balls [8] being larger than in the loose balls of [26]. We also investigated the sensitivity of our model to the form of the ridge energy (taking it as $\phi^{7/3}$ or changing the power $7/3$ to 1.1 and 5) or the

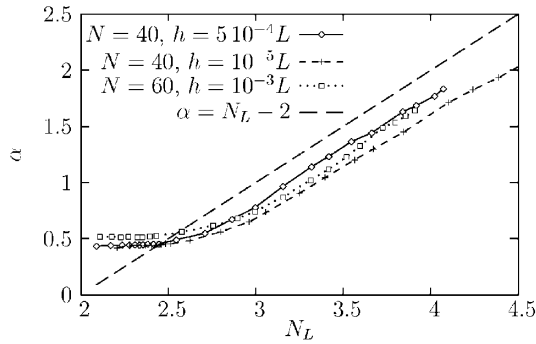


FIG. 4. The parameter α of the Gamma distribution P_{Γ} [Eq. (3)] as a function of the number of layers N_L [defined as the mean number of intersections of a half-infinite straight line with the system (see Fig. 1), averaged over all directions of the line and all samples].

confining potential (choosing a sharper r^6 which is closer to the confinement by a box). The exponents of the approach to jamming do not vary significantly within the error bars; the statistical properties are surprisingly robust. This robustness indicates that the system properties are determined mostly by geometry. As a consequence, our main point—the statistical properties change at large confinement because of stacking—should be insensitive to any other physical details. For instance, including plastic effects does not seem to be necessary, as the ridges stop moving when jamming is approached. Obviously, our results call for experimental measurements of length distributions at high confinement. Besides, our model can be refined in two ways. First, the number of vertices could be made free but one needs to find a criterion to add or suppress vertices. Second, it could be extended to 2D, which is more realistic but yields a more difficult numerical task: if the sheet is assumed to be a polyhedron, developability becomes a local constraint at each vertex whereas self-avoidance becomes even more computationally expensive.

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Note added in proof.—S. Chaieb brought to our attention Ref. [35], which opens the way to statistical studies based on full numerical simulations. Gamma laws should provide a good fit to the distributions of lengths obtained by the authors.

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- [1] U. Frisch, *Turbulence* (Cambridge University Press, Cambridge, England, 1995).
 [2] M. A. F. Gomes, J. Phys. A **20**, L283 (1987).
 [3] M. A. F. Gomes, G. L. Vasconcelos, and C. C. Nascimento, J. Phys. A **20**, L1167 (1987).

- [4] M. A. F. Gomes, T. I. Jyh, and T. I. Ren, J. Phys. A **23**, L1281 (1990).
 [5] *Statistical Mechanics of Membranes and Surfaces*, edited by D. Nelson, T. Piran, and S. Weinberg, Jerusalem Winter School for Theoretical Physics Vol. 5 (World Scientific, Singapore, 1988).
 [6] E. M. Kramer and T. A. Witten, Phys. Rep. **344**, 255 (2001).
 [7] A. Tzschichholz, A. Hansen, and S. Roux, cond-mat/9507xx.
 [8] F. Plouraboué and S. Roux, Physica A (Amsterdam) **227**, 173 (1996).
 [9] L. Landau and E. Lifchitz, *Théorie de l'élasticité* (Mir, Moscow, 1990).
 [10] M. Ben Amar and Y. Pomeau, Proc. R. Soc. A **453**, 729 (1997).
 [11] T. A. Witten and H. Li, Europhys. Lett. **23**, No. 1, 51 (1993).
 [12] A. Lobkovsky, S. Gentges, H. Li, D. Morse, and T. A. Witten, Science **270**, 1482 (1995).
 [13] A. E. Lobkovsky, Phys. Rev. E **53**, 3750 (1996).
 [14] E. Cerda, S. Chaieb, F. Melo, and L. Mahadevan, Nature (London) **401**, 46 (1999).
 [15] T. Mora and A. Boudaoud, Europhys. Lett. **59**, 41 (2002).
 [16] T. Liang and T. A. Witten, Phys. Rev. E **71**, 016612 (2005).
 [17] S. Chaieb and F. Melo, Phys. Rev. E **56**, 4736 (1997).
 [18] A. E. Lobkovsky and T. A. Witten, Phys. Rev. E **55**, 1577 (1997).
 [19] L. Pauchard and S. Rica, Philos. Mag. B **78**, 225 (1998).
 [20] A. Boudaoud, P. Patricio, Y. Couder, and M. B. Amar, Nature (London) **407**, 718 (2000).
 [21] A. V. Pogorelov, *Bendings of Surfaces and Stability of Shells*, Translation of Mathematical Monographs No. 72 (American Mathematical Society, Providence, 1988).
 [22] Y. Pomeau and S. Rica, C.R. Acad. Sci. Paris Série II **325**, 181 (1997).
 [23] K. Matan, R. B. Williams, T. A. Witten, and S. R. Nagel, Phys. Rev. Lett. **88**, 076101 (2002).
 [24] E. M. Kramer and A. E. Lobkovsky, Phys. Rev. E **53**, 1465 (1996).
 [25] P. A. Houle and J. P. Sethna, Phys. Rev. E **54**, 278 (1996).
 [26] D. L. Blair and A. Kudrolli, Phys. Rev. Lett. **94**, 166107 (2005).
 [27] A. J. Wood, Physica A (Amsterdam) **313**, 83 (2002).
 [28] P. Patricio and W. Krauth, Int. J. Mod. Phys., C **8**, 427 (1997).
 [29] E. M. Kramer and T. A. Witten, Phys. Rev. Lett. **78**, 1303 (1997).
 [30] J. A. Åström, J. Timonen, and M. Karttunen, Phys. Rev. Lett. **93**, 244301 (2004).
 [31] E. Villermaux and J. Duplat, Phys. Rev. Lett. **91**, 184501 (2003).
 [32] E. Villermaux, P. Marmottant, and J. Duplat, Phys. Rev. Lett. **92**, 074501 (2004).
 [33] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in C* (Cambridge University Press, Cambridge, England, 2002).
 [34] W. Feller, *An Introduction to Probability Theory and Its Applications* (John Wiley & Sons, New York, 1970).
 [35] G. A. Vliegthart and G. Gomper, Nat. Mater. **5**, 216 (2006).