Second-Sound Spectroscopy of a Nonequilibrium Superfluid-Normal Interface

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An experiment is proposed to test a previously developed theory of the hydrodynamics of a nonequilibrium heat current-induced superfluid-normal interface. It is shown that the interfacial "trapped" second-sound mode predicted by the theory leads to a sharp resonant dip in the reflected signal from an external second-sound pulse propagated toward the interface when its horizontal phase speed matches that of the interface mode. The influence of the interface on thermal fluctuations in the bulk superfluid is shown to lead to slow power dependence of the order parameter, and other quantities, on distance from it.

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In a seminal paper, Onuki [1] showed that when a uniform heat current $\mathbf{Q} = Q\hat{\mathbf{z}}$ is driven through a sample of ^{4}He very close to the superfluid transition, a situation can occur where the induced temperature profile divides the system into separate upstream ($z < 0$) normal and downstream ($z > 0$) superfluid regions. Within the interface between them (taken as centered on $z = 0$), the primary mode of heat transport converts from thermal diffusion, with temperature gradient $\nabla T = -\mathbf{Q}/\kappa$ where κ is the thermal conductivity (diverging at the superfluid transition), to superfluid counterflow, with a supercurrent $\mathbf{j}_s$ flowing toward the interface, and normal current $\mathbf{j}_n = -\mathbf{j}_s \propto \mathbf{Q}$ carrying the heat away from it. The latter flow shorts out all temperature gradients, leading to an asymptotically constant temperature $T_\infty(Q)$ deep on the superfluid side, $z \to \infty$ [2]. The interface width, $\xi(Q)$, diverges as $Q \to 0$, and plays the role of the fundamental correlation length in the system. Correspondingly, the order parameter ψ vanishes in the normal fluid, smoothly turns on through the interface, and takes the helical form $\psi(z) = |\psi_\infty(Q)|e^{-ik_\infty(Q)z}$ deep in the superfluid, where the phase gradient $k_\infty = -mv_s/\hbar$ is proportional to the superfluid velocity. A mean field calculation of these profiles is shown in Fig. 1.

This initial work spawned a series of experimental [3] and theoretical [4] investigations into this, and related, nonequilibrium superfluid critical phenomena [5]. Most relevant to the present work, in Ref. [6] the *dynamics* of the interface under external forcing was considered, and it was predicted that an interfacial second-sound mode exists, in which perturbations travel along the interface as waves with a well-defined sound speed $c(Q)$, and higher order damping constant $D(Q)$. The waves are trapped in the sense that their amplitude dies exponentially into the bulk phases on either side [6]. In this Letter an experiment is proposed, and the underlying theory developed, to verify the existence of this mode via second-sound scattering. It is shown that when a pulse is propagated toward the interface from the superfluid phase (see Fig. 2), strong resonant absorption occurs when its horizontal phase speed matches $c(Q)$, leading to a sharp dip, with depth scaling with $D(Q)$, in the reflected amplitude (Fig. 3 below). In a related effect, it is shown that thermal excitation of these same modes leads to slow power law corrections in $1/z$ to the order parameter and other quantities.

The analysis is based on the Model F equations of Halperin and Hohenberg [7]. These are derived from the Hamiltonian

$$\mathcal{H} = \int d^d r\left\{\frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}r_0|\psi|^2 + u_0|\psi|^4 + \frac{1}{2}\chi_0^{-1}(m - \chi_0 h_0 + \gamma_0\chi_0|\psi|^2)^2\right\} \tag{1}$$

where $\psi(\mathbf{r}, t)$ is the superfluid order parameter, $m(\mathbf{r}, t)$ is a linear combination of mass and energy density, and the last term provides the crucial coupling between the two. The equations of motion are obtained from

$$\begin{aligned}\partial_t\psi &= -2\Gamma_0\frac{\delta\mathcal{H}}{\delta\psi^*} + ig_0\psi\frac{\delta\mathcal{H}}{\delta m} + \theta_\psi,\\ \partial_t m &= \lambda_0\nabla^2\frac{\delta\mathcal{H}}{\delta m} - 2g_0\,\mathrm{Im}\left(\psi^*\frac{\delta\mathcal{H}}{\delta\psi^*}\right) + W + \theta_m,\end{aligned} \tag{2}$$

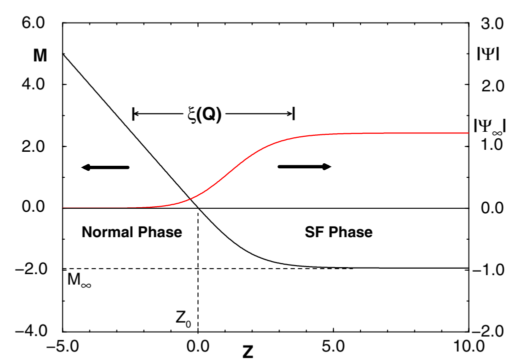

FIG. 1 (color online). Scaled mean field steady state temperature, M, and order parameter magnitude, $|\Psi|$, profiles through the interface as derived from (3), using parameters $a_0 = b_0 = 1$, $d_0 = 2$, and $c_0 = 0$, which produce $M_\infty = -1.936$, $|\Psi_\infty| = 1.216$, and $K_\infty = 0.676$.

where θ_ψ and θ_m are thermal white noise sources and $W = Q[\delta(z - z_1) - \delta(z - z_2)]$ provides a source of heat at $z_1 \to -\infty$ and a sink of heat at $z_2 \to +\infty$. The local temperature is defined as $\mu(\mathbf{r}, t) = \delta\mathcal{H}/\delta m(\mathbf{r}, t) = \chi_0^{-1} m + \gamma_0|\psi|^2 - h_0$. The various constant parameters may be partially determined by fits to experimental data [1,4]. The basic (mean field) length in the problem is $l_0 = (\lambda_0/2\chi_0\gamma_0 Q)^{1/3}$, and it is convenient to define rescaled time and space variables $\mathbf{R} = \mathbf{r}/l_0$ and $\tau = t/t_0$, with $t_0 = l_0^2/\mathrm{Re}\Gamma_0$. The equations of motion take the form

$$\begin{aligned} \partial_\tau\Psi &= -(1 + ic_0)[-\nabla_\mathbf{R}^2 + M + |\Psi|^2]\Psi + ia_0(M - M_\infty)\Psi + \Theta_\Psi, \\ (b_0/2a_0)\partial_\tau[d_0 M - |\Psi|^2] &= \nabla_\mathbf{R}^2 M + b_0\nabla_\mathbf{R}\cdot\mathbf{J} + \delta(Z - Z_1) - \delta(Z - Z_2) + \Theta_M, \end{aligned} \tag{3}$$

where $a_0 = g_0/2\gamma_0\chi_0(\mathrm{Re}\Gamma_0)$, $b_0 = \gamma_0\chi_0 g_0/2\lambda_0 u_0$, $c_0 = \mathrm{Im}\Gamma_0/\mathrm{Re}\Gamma_0$, $d_0 = 2u_0/\chi_0\gamma_0^2$, $\Psi = 2l_0\sqrt{u_0}e^{-ig_0\tilde{\mu}t}\psi$, $M = [r_0 + 2\gamma_0\chi_0\mu]l_0^2$, $\tilde{\mu} = \lim_{z\to\infty}\mu(z)$ is the asymptotic superfluid temperature, $M_\infty = \lim_{Z\to\infty}M(Z)$, and $\mathbf{J} = \mathrm{Im}(\Psi^*\nabla_\mathbf{R}\Psi)$ is the supercurrent density. The rescaled thermal noise sources $\Theta_\Psi = (2\sqrt{u_0}l_0^3/\mathrm{Re}\Gamma_0)e^{-ig_0\tilde{\mu}t}\theta_\psi$ and $\Theta_M = (2\gamma_0\chi_0 l_0^4/\lambda_0)\theta_m$ have covariances that diverge as $l_0^{4-d} \propto Q_0^{(d-4)/3}$. An expansion in $4 - d$ is then required for a full theory in the small Q critical regime [4], but is difficult for the full interface problem. However, the experiment proposed here is less concerned with subtleties of nonequilibrium criticality, than with basic hydrodynamical properties of the interface that are more sharply defined and easier to investigate at larger Q. A correspondingly simpler theoretical approach will be taken in which Q is assumed large enough that the noise may be treated as a perturbation on the homogeneous equations [8].

Writing $\Psi = |\Psi|e^{i\phi}$ and $\mathbf{U} = (|\Psi|, \phi, M)$, one may readily obtain a noise free steady state solution with an interface (see Fig. 1), denoted by $\mathbf{U}_0(Z)$, with $M_0(Z) = -Z$, $Z \to -\infty$, and $\mathbf{U}_0(Z) \to (|\Psi_\infty|, -K_\infty Z, M_\infty)$, $Z \to +\infty$, with the constraints $K_\infty^2 + |\Psi_\infty|^2 + M_\infty = 0$ and $b_0 K_\infty|\Psi_\infty|^2 = 1$.

Long wavelength, harmonic excitations of the interface are solutions of the form $\delta\mathbf{U} = \mathbf{U}(\mathbf{R}, \tau) - \mathbf{U}_0 = \delta\mathbf{U}_{\mathbf{q},\omega}(Z)e^{i\mathbf{q}\cdot(X,Y) - i\omega\tau}$, and may be thought of as being driven by an external source on the plane $Z = Z_2$ with

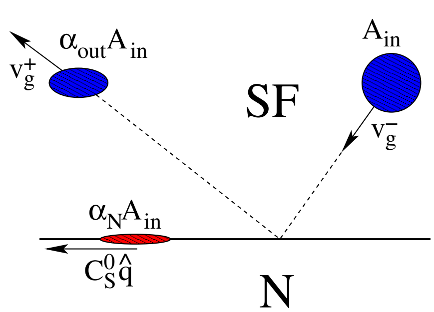

FIG. 2 (color online). Scattering geometry: An incoming pulse with amplitude A_{in} and spectrum narrowly centered on horizontal wave vector $\mathbf{q}$ and frequency ω approaches the interface at velocity $\mathbf{v}_g^-$. The reflected pulse, with relative amplitude α_{out}, moves away at velocity $\mathbf{v}_g^+$, accompanied by an excitation of the interface itself with relative amplitude α_N moving at speed C_S^0 along the direction of $\mathbf{q}$. When the resonance condition $\omega/q = C_S^0$ holds, $\alpha_{\mathrm{out}} \simeq 0$ (see Fig. 3) and the pulse is strongly absorbed by the interface.

fixed frequency ω and transverse wave vector $\mathbf{q}$. These satisfy linearized equations

$$i\omega\hat{\mathbf{L}}_3\delta\mathbf{U}_{\mathbf{q},\omega}(Z) = [\hat{\mathbf{L}}_0 + \hat{\mathbf{L}}_1\partial_Z + \hat{\mathbf{L}}_2(\partial_Z^2 - q^2)]\delta\mathbf{U}_{\mathbf{q},\omega}(Z), \tag{4}$$

with Z-dependent matrix coefficients which follow by straightforward linearization of (3) about $\mathbf{U}_0$, but whose exact form will not be required here. Deep on the superfluid side, the solution is a sum of incoming and outgoing scattered waves,

$$\delta\mathbf{U}_{\mathbf{q},\omega}(Z) = A_{\mathrm{in}}[\mathbf{V}_{\mathbf{q},\omega}^- e^{iq_z^- Z} + \alpha_{\mathrm{out}}\mathbf{V}_{\mathbf{q},\omega}^+ e^{iq_z^+ Z}], \tag{5}$$

obtained from the asymptotic form of (4) with the replacement $\partial_Z \to iq_z$. Here A_{in} is the amplitude of the incoming excitation, and $\alpha_{\mathrm{out}}(\mathbf{q}, \omega)$ is the relative amplitude of the reflected wave. The wave vector components $q_z^+(\mathbf{q}, \omega) > q_z^-(\mathbf{q}, \omega)$ are the two solutions to the second-sound disper-

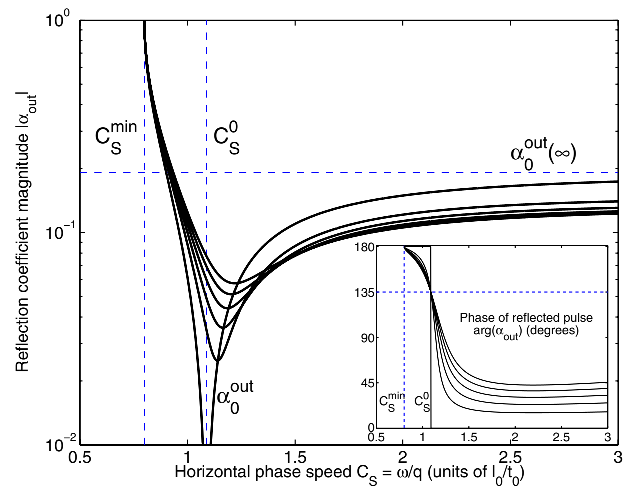

FIG. 3 (color online). Magnitude and phase (inset) of the reflection coefficient $\alpha_{\mathrm{out}} = \alpha_0^{\mathrm{out}} + \alpha_1^{\mathrm{out}}\sqrt{q}$ plotted as functions of the horizontal phase speed C_S for $q = 0, 0.1, 0.2, 0.3, 0.4, 0.5$, using scaled Model F parameters $a_0 = b_0 = 1$, $d_0 = 2$, $c_0 = 0$. In both figures, sharper curves correspond to smaller q. The minimum allowed phase speed $C_S^{\min}$ occurs for $q_z^\pm = q_{z,c}$, hence a purely horizontal group velocity. The large C_S asymptote of α_0^{out} corresponds to a high frequency pulse propagated nearly straight at the interface. For small q the minimum occurs at $C_S = C_{S,0} + D_{S,0}\sqrt{q}$ with value $B_0 D_{S,0}\sqrt{q}$, $B_0 = 2a_0^2 M_\infty^2(|\Psi_\infty|^2 - 2K_\infty^2)/9K_\infty^2|\Psi_\infty|^2(C_S^0)^3$, scaling with the interfacial dissipation constant. The phase switches rapidly (instantaneously for $q \to 0$) between near-π to near-zero as C_S increases through C_S^0.

sion relation

$$0 = (1 + d_0)\omega^2 + 4a_0 K_\infty q_z \omega - 2a_0^2|\Psi_\infty|^2 q^2 - 2a_0^2(|\Psi_\infty|^2 - 2K_\infty^2)q_z^2 + O(\omega^3, q_z^3, q_z q^2, \ldots), \quad (6)$$

where stability requires that $|\Psi_\infty|^2 > 2K_\infty$ [1]. The corresponding eigenvectors are normalized to have unit ϕ-component, $\mathbf{V}^\pm_{\mathbf{q},\omega} = (i(\omega + 2a_0 K_\infty q_z^\pm)/2a_0|\Psi_\infty|, 1, -i\omega/a_0)$ with $O(\omega^2, (q_z^\pm)^2, q^2, \ldots)$ corrections in the first and last components. To the exhibited order, (6) may be put in the elliptical form $(q_z - q_{z,c})^2/(\Delta q_z)^2 + q^2/(\Delta q)^2 = 1$, with ellipse center, and semimajor axes given by $q_{z,c} = K_\infty\omega/a_0(|\Psi_\infty|^2 - 2K_\infty^2)$, $\Delta q_z = q_{z,c}\sqrt{(1 + d_0)(|\Psi_\infty|^2/2K_\infty^2 - 1) + 1}$, $\Delta q = \Delta q_z\sqrt{1 - 2K_\infty^2/|\Psi_\infty|^2}$. The direction of pulse propagation is determined by the group velocity

$$\mathbf{v}_g(\mathbf{q}, \omega) = \frac{\omega}{1 + q_{z,c}(q_z - q_{z,c})/\Delta q_z^2}\left(\frac{\mathbf{q}}{(\Delta q)^2}, \frac{q_z - q_{z,c}}{(\Delta q_z)^2}\right), \quad (7)$$

so that $v^+_{g,z} > 0 > v^-_{g,z}$ indeed have opposite sign, and for each $\mathbf{q}$, ω one may form pulses propagating towards and away from the interface (see Fig. 2).

The dissipative normal phase dynamics implies an exponentially decaying asymptote $\delta M_{\mathbf{q},\omega} = A_{\rm in}\alpha_N(\mathbf{q}, \omega)e^{iq_z^N Z}$, $Z \ll 0$, with ${\rm Im} q_z^N < 0$ determined by the diffusion relation, $i(b_0 d_0/2a_0)\omega = q^2 + (q_z^N)^2$. In solving (4), we will formally treat ω, q as small parameters of the same order, thus viewing $C_S \equiv \omega/q$ as a fixed $O(1)$ parameter. It follows that $q_z^\pm = q\sum_{k=0}^\infty \pi_k^\pm q^{2k}$, $q_z^N = q^{1/2}\sum_{k=0}^\infty \pi_k^N q^k$ with known coefficients, $\pi_0^N = -(1 + i)\sqrt{b_0 d_0 C_S/4a_0}$, $\pi_0^\pm = \pi_0^S \pm \Delta\pi_0^S\sqrt{1 - 1/(\Delta C)^2}$, where the ratios $\pi_0^S = q_{z,c}/q$, $\Delta\pi_0^S = \Delta q_z/q$, $\Delta C = \Delta q/q$ are functions only of C_S.

The appearance of $q^{1/2}$ in the normal phase leads one to expect the solution to (4) to have an expansion in $q^{1/2}$ rather than q or q^2: $\delta\mathbf{U}_{\mathbf{q},\omega}(Z) = \sum_{k=0}^\infty q^{k/2}\delta\mathbf{U}_k(Z)$, $\alpha_{{\rm out},N} = \sum_{k=0}^\infty \alpha_k^{{\rm out},N} q^{k/2}$. Asymptotic matching will produce simultaneous expansions in powers of $q^{1/2}$ and Z (coming from the expansion of the exponentials) on each side of the interface, and matching each monomial coefficient will allow determination of the unknown coefficients $\alpha_k^{{\rm out},N}$. Defining $\mathbf{L}_Z = \mathbf{L}_0 + \mathbf{L}_1\partial_Z + \mathbf{L}_2\partial_Z^2$, substitution of this expansion into (4) leads to the sequence of relations

$$\begin{gathered} \mathbf{L}_Z\delta\mathbf{U}_0 = 0, \qquad \mathbf{L}_Z\delta\mathbf{U}_1 = 0, \\ \mathbf{L}_Z\delta\mathbf{U}_2 = iC_S\mathbf{L}_3\delta\mathbf{U}_0, \qquad \mathbf{L}_Z\delta\mathbf{U}_3 = iC_S\mathbf{L}_3\delta\mathbf{U}_1, \quad (8) \\ \mathbf{L}_Z\delta\mathbf{U}_4 = iC_S\mathbf{L}_3\delta\mathbf{U}_2 - \mathbf{L}_2\delta\mathbf{U}_0, \end{gathered}$$

etc. Therefore, $\delta\mathbf{U}_{0,1}$ correspond to zero frequency perturbations of the interface, i.e., infinitesimal motions that simply produce a new steady state. There are two of these: a global phase rotation $\mathbf{v}_0 = (0, -K_\infty, 0)$ (normalized using K_∞ for later convenience), and a uniform translation of the interface $\mathbf{v}_1 = \partial_Z\mathbf{U}_0$. The asymptotes are $\mathbf{v}_1 \to \mathbf{v}_0$ for $Z \gg 0$, $\delta M_1 \to -1$ for $Z \ll 0$. The appearance of $\mathbf{L}_Z\delta\mathbf{U}_k$ in (8) at each order means that the solution can be determined only up to an arbitrary linear combination $\mu_k\mathbf{v}_0 + \nu_k\mathbf{v}_1$ whose coefficients must be determined from the matching conditions.

To begin, one obtains $\delta\mathbf{U}_{0,1} = \mu_{0,1}\mathbf{v}_0 + \nu_{0,1}\mathbf{v}_1$. The $O(q^0, q^{1/2})$ matching conditions yield $\nu_0 = -\alpha_0^N = 0$, $\nu_1 = -\alpha_1^N$ on the normal side, and $1 + \alpha_0^{\rm out} = -K_\infty\mu_0$, $\alpha_1^{\rm out} = -K_\infty(\mu_1 + \nu_1)$ on the superfluid side.

At $O(q)$ one obtains $\delta\mathbf{U}_2 = i\mu_0 C_s\mathbf{v}_2 + \mu_2\mathbf{v}_0 + \nu_2\mathbf{v}_1$, in which $\mathbf{v}_2$ satisfies $\mathbf{L}_Z\mathbf{v}_2 = \mathbf{L}_3\mathbf{v}_0$. It follows that $\mathbf{v}_2$ is the change in shape of the interface under a perturbation of the heat current, $Q \to 1 + \delta Q$, and results from compression and rarefaction of the heat current in the vicinity of the interface due to the incoming wave. Adjusting Q leads to a simple rescaling of $\mathbf{U}_0$, and one obtains exact the result $\mathbf{v}_2 = (K_\infty/2a_0 M_\infty)[Z\mathbf{v}_1 + (|\Psi_0|, 0, 2M_0)]$ [6], with asymptotes $\mathbf{v}_2 \to (K_\infty/2a_0 M_\infty)(|\Psi_\infty|, -K_\infty Z, 2M_\infty)$, $Z \gg 0$, and $M_2 \to -3K_\infty Z/2a_0 M_\infty$, $Z \ll 0$. The most important matching condition now arises from the term linear in Z on the superfluid side, which produces the additional constraint $\alpha_0^{\rm out}\pi_0^+ + \pi_0^- = -\mu_0 C_S K_\infty^2/2a_0 M_\infty$, using which one obtains,

$$\alpha_0^{\rm out} = -(1 + K_\infty\mu_0) = -\frac{\pi_0^- - K_\infty C_S/2a_0 M_\infty}{\pi_0^+ - K_\infty C_S/2a_0 M_\infty}, \quad (9)$$

along with $\nu_2 = -\alpha_2^N$, and $\nu_1 = -\alpha_1^N = 3K_\infty\mu_0 C_s/2a_0 M_\infty\pi_0^N$. The latter corresponds to the actual spatial amplitude of the interface motion, $\delta Z(\mathbf{q}, \omega) = A_{\rm in}\nu_1 q^{1/2}$. The numerator in (9) vanishes, implying full absorption of the incoming wave, for $C_S = C_{S,0}$, with

$$C_{S,0}^2 = \frac{4a_0^2|\Psi_\infty|^2 M_\infty^2}{2d_0 M_\infty^2 + |\Psi_\infty|^2(2|\Psi_\infty|^2 - K_\infty^2)}, \quad (10)$$

corresponding precisely to (the square of) the interfacial sound speed (with physical value $c = C_{S,0}l_0/\tau_0$) found in Ref. [6].

At order $O(q^{3/2})$ one obtains $\delta\mathbf{U}_3 = iC_s(\mu_1\mathbf{v}_2 + \nu_1\mathbf{v}_3) + \mu_3\mathbf{v}_0 + \nu_3\mathbf{v}_1$, in which $\mathbf{v}_3$ satisfies $\mathbf{L}_Z\mathbf{v}_3 = \mathbf{L}_3\mathbf{v}_1$. It follows that $\mathbf{v}_3$ is the perturbation to the interface profile induced by a change δQ of the incoming heat current on the normal side *only*, with that exiting on the superfluid side unchanged: $\mathbf{U}(Z, \tau) - \mathbf{U}_0(Z - v\tau) \propto v\mathbf{v}_3(Z - v\tau)$. This causes heat to build up behind the interface, moving it forward at an instantaneous speed $v \propto \delta Q/Z_1$. This heating effect leads to dissipation of the interface motion, which is *singularly damped* [6], at rate $\propto q^{3/2}$ rather than the bulk q^2 [which would arise from corrections to (6)].

Although an analytic form for $\mathbf{v}_3$ is not available, it can be obtained numerically. For the purposes of determining μ_1, ν_2, α_2^N, $\alpha_1^{\rm out}$, only two numbers are required, namely, the coefficients k_3, m_3 in the asymptotic forms $v_{3,\phi} \to k_3 Z$, $Z \gg 0$, and $v_{3,M} \to (b_0 d_0/4a_0)Z^2 + m_3 Z$, $Z \ll 0$ (there may also be constant terms, but these may be absorbed into μ_3, ν_3, which remain undetermined at this order). In addition to the $O(q)$ constraint $\alpha_1^{\rm out} = -K_\infty(\mu_1 + \nu_1)$, one obtains from the matching:

$$(\pi_0^N/C_S)\alpha_2^N + (3K_\infty/2a_0M_\infty)\mu_1 = \nu_1 m_3,$$
$$(\pi_0^+/C_S)\alpha_1^{\rm out} + (K_\infty^2/2a_0M_\infty)\mu_1 = \nu_1 k_3, \tag{11}$$

with solution

$$\alpha_1^{\rm out} = -\frac{3C_S^2(\pi_0^+ - \pi_0^-)}{2a_0M_\infty\pi_0^N}\frac{k_3 + K_\infty^2/2a_0M_\infty}{(\pi_0^+ - K_\infty C_S/2a_0M_\infty)^2},$$
$$\alpha_2^N = -\nu_2 = \frac{C_S}{\pi_0^N}\left[m_3\nu_1 + \frac{3(\alpha_1^{\rm out} + K_\infty\nu_1)}{2a_0M_\infty}\right]. \tag{12}$$

Since π_0^N is complex, the zero of the first order combination $\alpha_{\rm out}(q) \approx \alpha_0^{\rm out} + q^{1/2}\alpha_1^{\rm out}$ is now shifted to the complex value $\omega = C_{S,0}q + (1-i)D_{S,0}q^{3/2} + O(q^2)$, with dissipation parameter

$$D_{S,0} = \frac{9K_\infty(C_S^0)^{7/2}(k_3 + K_\infty^2/2a_0M_\infty)}{4\sqrt{a_0^3b_0d_0}M_\infty^2}. \tag{13}$$

and corresponding physical value $D = D_S^0 l_0^{3/2}/\tau_0$.

Using the parameters $a_0 = b_0 = 1$, $d_0 = 2$, $c_0 = 0$, one obtains $C_{S,0} = 1.089$, $k_3 = 0.488$, $D_{S,0} = 0.143$, and $m_3 = 0.637$ [6]. The resulting magnitude and phase of the reflection coefficient $\alpha_{\rm out}$, plotted as a function of C_S for various q, are shown in Fig. 3.

Finally, consider the effects of thermal noise, under the assumption, as discussed above, that the scaled noise amplitude is sufficiently small that it may be treated within the linear response regime. Using the mode decomposition $\mathbf{U} = \sum_{\mathbf{q},\omega}A_{\mathbf{q},\omega}(\tau)\delta\mathbf{U}_{\mathbf{q},\omega}(Z)e^{i\mathbf{q}\cdot(X,Y)}$, one obtains an equation of motion for the amplitude

$$(\partial_\tau + i\omega)A_{\mathbf{q}\omega}(\tau) = \theta_{\mathbf{q},\omega}(\tau), \tag{14}$$

in which the white noise $\theta_{\mathbf{q},\omega}(\tau)$ is the appropriate projection of Θ_Ψ, Θ_M onto the mode eigenvector. The solution

$$A_{\mathbf{q},\omega}(\tau) = \int_{-\infty}^{\tau} d\tau' e^{-i\omega(\tau-\tau')}\theta_{\mathbf{q},\omega}(\tau') \tag{15}$$

(where the small dissipative negative imaginary part that gets added to ω at order q^2 is actually required here to ensure convergence), inserted back into $\mathbf{U}$, allows one to compute stochastic averages of various quantities (calculations are tedious and badly encumbered by matrix indices, and will be presented in detail elsewhere). Not surprisingly, it is the phase correlations that are the most important, decaying at long distances as a power law $\langle\delta\phi(\mathbf{R})\delta\phi(\mathbf{R}')\rangle \propto |\mathbf{R} - \mathbf{R}'|^{2-d}$ (with $d = 3$ here) deep in the superfluid phase, with a complicated anisotropic coefficient depending on the angle between $\mathbf{R} - \mathbf{R}'$ and the heat flow direction $\hat{\mathbf{z}}$. The extra $O(q)$ factors in the $|\Psi|$, M components of $\mathbf{V}_{\mathbf{q},\omega}^{\pm}$ produce weaker power laws $\langle\delta|\Psi(\mathbf{R})|\delta|\Psi(\mathbf{R}')|\rangle$, $\langle\delta M(\mathbf{R})\delta M(\mathbf{R}')\rangle \propto |\mathbf{R} - \mathbf{R}'|^{-d}$, also with anisotropic coefficients.

Closer to the interface, there are residual correlations, coming from interference between the incoming and reflected waves in $\mathbf{V}_{\mathbf{q},\omega}^{\pm}$, that produce Z dependence in the local fluctuations: $\langle\delta\phi(\infty)^2\rangle - \langle\delta\phi(\mathbf{R})^2\rangle \propto 1/Z^{d-2}$, $\langle\delta M(\infty)^2\rangle - \langle\delta M(\mathbf{R})^2\rangle$, $\langle\delta|\Psi(\infty)|^2 - \delta|\Psi(\mathbf{R})|^2\rangle \propto 1/Z^d$, where the argument ∞ is shorthand for $Z \to \infty$. The phase fluctuations have a very strong effect on the complex order parameter:

$$\langle\Psi(\mathbf{R})\rangle \approx \langle|\Psi(\mathbf{R})|\rangle\langle e^{i\phi(\mathbf{R})}\rangle = \Psi_0(\mathbf{R})e^{-(1/2)\langle\delta\phi(\mathbf{R})^2\rangle}$$
$$\approx \Psi_0(\mathbf{R})e^{-(1/2)\langle\delta\phi(\infty)^2\rangle}(1 - BZ^{2-d}), \tag{16}$$

where B is a coefficient, and the Gaussian property of the noise has been used to average the exponential. There are two interesting effects here: First, phase fluctuations can significantly reduce the magnitude of the order parameter even in the linear response regime [validity of (16) requires only the weaker assumption that the space-time derivatives of $\delta\phi$, as opposed to $\delta\phi$ itself, be small]. Second, the presence of the interface induces a slow power law approach of $|\langle\Psi(\mathbf{R})\rangle|$ to its asymptotic superfluid value, contrasting with the exponential approach of $\Psi_0(\mathbf{R})$. This power law is not induced by the positional fluctuations of the interface itself, which remain strongly bounded [6], but by the effects of its mere presence as a reflecting boundary on the long-range bulk superfluid correlations.

At linear order, the average temperature $\langle M(\mathbf{R})\rangle = M_0(\mathbf{R})$ remains equal to its mean field value, with the power law visible only in the variance. However, it is likely that similar power laws will be induced in $\langle M(\mathbf{R})\rangle$ if nonlinear corrections are taken into account.

[1] A. Onuki, J. Low Temp. Phys. **50**, 433 (1983).

[2] Thermal noise actually induces the occasional expanding vortex ring, giving rise to bulk dissipation, and hence a small temperature gradient, making $T_\infty(Q)$ less sharply defined, but this is a higher order effect that will be ignored in the present work.

[3] See, e.g., R. V. Duncan, G. Ahlers, and V. Steinberg, Phys. Rev. Lett. **60**, 1522 (1988); W. A. Moeur, P. K. Day, F.-C. Liu, S. T. P. Boyd, M. J. Adriaans, and R. V. Duncan, Phys. Rev. Lett. **78**, 2421 (1997).

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