Centrifugal Forces Alter Streamline Topology and Greatly Enhance the Rate of Heat and Mass Transfer from Neutrally Buoyant Particles to a Shear Flow

G. Subramanian and D. L. Koch

School of Chemical and Bio-Molecular Engineering, Cornell University, Ithaca, New York 14853, USA

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Centrifugal forces break the degenerate closed-streamline configuration that occurs in simple shear flow past a neutrally buoyant torque-free particle in the inertialess limit. The broken symmetry allows heat or mass to be convected away in an efficient manner in sharp contrast to the inertialess diffusion-limited scenario. The dimensionless transfer rate, characterized by the Nusselt number, is found to be $\mathrm{Nu} = 0.33(\mathrm{Re}\,\mathrm{Pe})^{1/3} + O(1)$ for small but finite Re when $\mathrm{Re}\,\mathrm{Pe} \gg 1$. Here, the particle Reynolds number (Re) is a dimensionless measure of the inertial forces, while the Peclet number (Pe) measures the relative importance of the convective and the diffusive transfer mechanisms. The symmetry-breaking bifurcation is expected to occur in generic shearing flows, and represents a possible means for heat or mass transfer enhancement from the dispersed phase in multiphase systems.

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Much insight has been gained into the flow behavior of suspensions in which viscous forces dominate over fluid inertia [1,2]. However, the fluid flow and heat and mass transfer within suspensions may change in fundamental ways when fluid inertia is important on the micro- (or particle-) length scale. In sheared suspensions, microscale inertia is characterized by the particle Reynolds number, $\mathrm{Re} = a^2\dot{\gamma}/\nu$, a dimensionless ratio of the magnitudes of inertial and viscous forces on the scale of a single particle; here, a is the particle size (radius for spherical particles), $\dot{\gamma}$ a measure of the fluid velocity gradient, and ν the kinematic viscosity of the suspending fluid. In this Letter, we show that microscale inertia dramatically alters the streamline topology and the heat or mass transfer in suspensions of neutrally buoyant particles subject to simple shear flow when Re is small but nonzero.

Inertial effects become important in suspensions of large particles (tens of microns) or in relatively low viscosity fluids. An ambient shear will dominate effects related to sedimentation when the density mismatch with the suspending medium is small. Applications include emulsion polymerization where the exothermic polymerization occurs in suspended droplets, bioreactors using suitably immobilized, cell or cell aggregates, and drug delivery with porous polymer particles.

Progress in the understanding of microscale inertial effects has been limited despite their fundamental significance. Analytical results exist in two dimensions for simplified scenarios; for instance, the flow around a cylinder at finite Re in an ambient shearing flow [3]. Solutions to similar problems in three dimensions have mostly remained elusive, since the presence of vortex stretching enormously complicates the fluid dynamics. We show here that the dimensionality of the embedding space plays a crucial role in heat or mass transport.

It has been known since the work of Acrivos and coworkers in the 1970's [3,4] that, in the absence of inertia ($\mathrm{Re} = 0$) the heat transfer from neutrally buoyant torque-free spheres in simple shear flow is diffusion limited at large Pe [5]; the Peclet number, $\mathrm{Pe} = aU/\alpha$, is a dimensionless measure of the relative dominance of convection and diffusion with U being the velocity scale ($U \sim \dot{\gamma}a$ in a shearing flow) and α the thermal diffusivity. The dominance of diffusive processes even at large Pe is unexpected, and stems from the existence of closed streamlines in a reference frame translating with the sphere. As a result, fluid elements near the sphere move around it in periodic orbits, being unable to carry heat away, and one does not observe the familiar boundary layer enhancement of heat transfer from a particle in open flows [6].

For uniform flow past a fixed sphere, the dimensionless flux from the particle surface, as characterized by the Nusselt number, is $\mathrm{Nu} \sim O(\mathrm{Pe}^{1/3})$ for large Pe. The Nusselt number is the heat flux measured in (dimensionless) conduction units; thus, $\mathrm{Nu} = Q/4\pi ka(T_s - T_\infty)$, where Q is the dimensional heat flux, k the thermal conductivity, and $(T_s - T_\infty)$ the temperature difference between the surface and ambient fluid. For large Pe, the heat diffuses across a thin $O(a\mathrm{Pe}^{-1/3})$ boundary layer on the particle surface, and is thereafter rapidly carried away by fluid elements spending only a $O[a^2/(\alpha\mathrm{Pe}^{2/3})]$ time in the neighborhood of the particle. As Pe increases, the heat transfer becomes increasingly efficient when compared to pure conduction. On the other hand, for a torque-free sphere in simple shear flow ($U \sim \dot{\gamma}a$) at $\mathrm{Re} = 0$, the region of closed streamlines around the sphere implies that the residence time of a fluid element in its vicinity diverges, and convection is no longer effective in enhancing heat transfer. In contrast to open flows, Nu tends to a $O(1)$ constant for large Pe; indeed, Acrivos [7] showed that $\lim_{\mathrm{Pe}\to\infty}\mathrm{Nu} \approx 4.5$ when $\mathrm{Re} = 0$.

The situation is, however, radically altered at small but finite Re via a bifurcation in the streamline topology resulting from inertial forces. The inertialess streamline configuration for a torque-free sphere in simple shear is shown in Fig. 1. In the absence of the sphere, an exact balance of the extensional and rotational components of the undisturbed simple shear leads to a rectilinear streamline pattern. The presence of the rotating sphere alters the local balance of extension and vorticity, and creates a rotational region with closed streamlines in its vicinity. This region of closed streamlines, in fact, has an infinite volume. An axisymmetric separatrix envelope, infinite in extent, separates the fore-aft symmetric open streamlines outside from closed ones within. The symmetry of the streamline configuration is rooted in the linearity and reversibility of the Stokes equations [2]. At finite Re fluid elements circulating around the sphere are centrifuged out, destroying the closed orbit structure present at $\mathrm{Re} = 0$. Streamlines close to the torque-free sphere, and in the plane of shear, now spiral outward. Incompressibility ensures that the net radial outflow that results is compensated by a corresponding influx of fluid along the vorticity axis. The finite Re velocity field also exhibits antisymmetric recirculating wakes (see Fig. 2) at distances along the flow axis greater than $O(a\mathrm{Re}^{-3/10})$ [8,9], these regions diminishing in extent away from the plane of shear. This wake structure, together with the outward spiraling close to the sphere, opens up channels via which heat can now be carried away by convection. The finite Re streamline pattern in Fig. 2 was determined using results for the $O(\mathrm{Re})$ velocity components obtained by Peery [10]; as discussed below, the topological implications of the $O(\mathrm{Re})$ inertial correction appear to have gone unnoticed. Note that the finite Re streamline topology is still invariant to a π rotation about the vorticity axis, and therefore consistent with the antisymmetry of simple shear.

The broken symmetry at finite Re allows for a net convection of heat away from the sphere, and one must observe a boundary layer enhancement of heat transfer at large enough Pe in a manner akin to open flows. However, although the Nusselt number is $O(\mathrm{Pe}^{1/3})$ for large Pe, there must be an additional dependence on Re because the residence time of a fluid element in the thermal boundary layer is a function of Re. The residence time evidently diverges at $\mathrm{Re} = 0$ when the streamlines in the vicinity of the sphere close, and is therefore a decreasing function of Re for small Re. As is shown below, in the limit $\mathrm{Re}\,\mathrm{Pe} \gg 1$, the dominant transfer of heat occurs across a thin $O(\mathrm{Re}\,\mathrm{Pe})^{-1/3}$ boundary layer, and it becomes possible to analytically determine Nu as a function of Re and Pe.

To begin with, we write down the equations governing the transport of momentum and heat. The velocity ($\boldsymbol{u}$) and pressure fields (p) satisfy the Navier-Stokes equations with the continuity equation enforcing the incompressibility constraint. In the dimensionless form, we have

$$\mathrm{Re}\,(\boldsymbol{u}\cdot\nabla\boldsymbol{u}) = -\nabla p + \nabla^2\boldsymbol{u}, \qquad \nabla\cdot\boldsymbol{u} = 0, \tag{1}$$

in a frame of reference that translates with the neutrally buoyant sphere [11]. The velocity field in this reference frame is steady and satisfies the following boundary con-

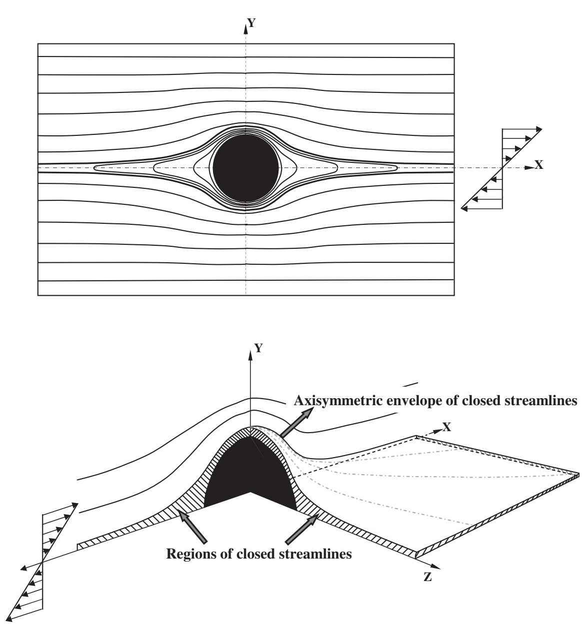

FIG. 1. The two figures show the fore-aft symmetric streamline pattern in the plane of shear, and the corresponding three-dimensional streamline topology, respectively, for a torque-free sphere in simple shear flow without inertia; an axisymmetric separatrix envelope separates the closed from the open streamlines. Here, x, y, and z denote the flow, gradient, and vorticity axes, respectively.

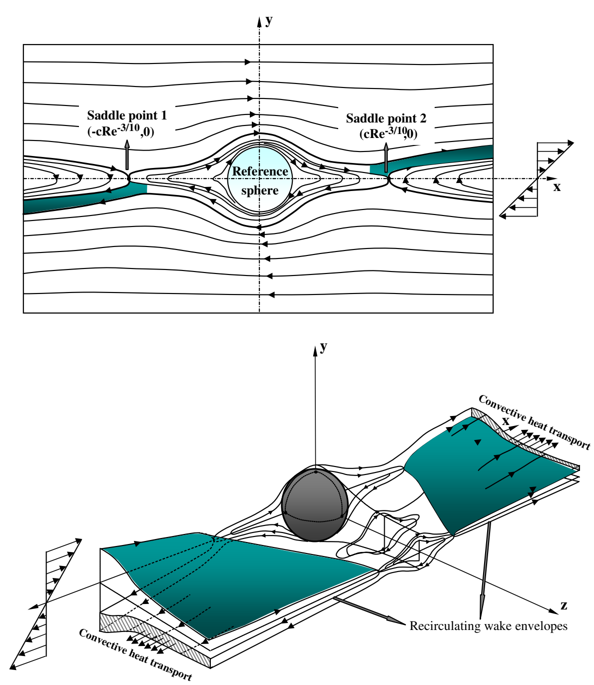

FIG. 2 (color online). The top figure shows the streamline pattern in the plane of shear, at finite Re, for a torque-free sphere in simple shear flow. The figure below depicts the finite Re three-dimensional streamline topology, including the convective channels that border the recirculating wakes.

ditions:

$$\boldsymbol{u} = \boldsymbol{\Omega}_s \wedge \boldsymbol{r} \quad \text{at} \quad r = 1, \tag{2}$$

$$\boldsymbol{u} \rightarrow \boldsymbol{\Gamma} \cdot \boldsymbol{r} \quad \text{as} \quad r \rightarrow \infty, \tag{3}$$

where $\boldsymbol{\Gamma} = \mathbf{1}_x \mathbf{1}_y$ is the transpose of the velocity gradient tensor for simple shear flow; here, x, y, and z correspond to the flow, gradient, and vorticity directions, respectively, of the ambient simple shear. It is convenient to define a normalized temperature $\Theta = (T - T_\infty)/(T_s - T_\infty)$, which satisfies a convection-diffusion equation,

$$\mathrm{Pe}\,(\boldsymbol{u} \cdot \boldsymbol{\nabla}\Theta) = \nabla^2 \Theta, \tag{4}$$

with the boundary conditions

$$\Theta = 1 \quad \text{at} \quad r = 1, \tag{5}$$

$$\Theta \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty. \tag{6}$$

The angular velocity of the sphere, Ω_s, in (2) is determined from the torque-free constraint; in mathematical terms, the condition may be expressed as

$$\int_{r=1} \boldsymbol{r} \wedge (\boldsymbol{\sigma} \cdot \boldsymbol{n}) dS = 0, \tag{7}$$

where $\boldsymbol{r} \wedge (\boldsymbol{\sigma} \cdot \boldsymbol{n})|_{r=1}$ is the antisymmetric first moment of the force density on the surface of the sphere, $\boldsymbol{\sigma} = -p\boldsymbol{\delta} + (\boldsymbol{\nabla u} + \boldsymbol{\nabla u}^\dagger)$ being the stress tensor.

We assume the temperature variations to be small enough for fluid properties to remain unaffected, as is typically the case. One may then first obtain the velocity field from solving (1), and thereafter, solve (4) for the temperature field using the boundary conditions (5) and (6). Finally, the dimensionless rate of heat transfer from the sphere surface may be evaluated as

$$\mathrm{Nu} = -\frac{1}{4\pi} \int_{r=1} \frac{\partial \Theta}{\partial r} dS. \tag{8}$$

For $\mathrm{Re} = 0$, (1) reduces to the Stokes equations which are easily solved. With the inertial terms included, the solution of (1) in an unbounded domain is a nontrivial task even in the limit $\mathrm{Re} \ll 1$, requiring singular perturbation techniques [8,12]. The difficulty arises because the inertial acceleration eventually becomes comparable to viscous forces at distances beyond an inertial screening length that for simple shear flow scales as $a\mathrm{Re}^{-1/2}$. Thus, for any finite Re, the Stokes velocity field is no longer a uniformly valid approximation. In the present context, however, the dominant temperature variations are expected to occur in a thin boundary layer on the surface of the sphere for large Pe. Therefore, one only requires the form of the velocity field, to $O(\mathrm{Re})$, near the sphere surface. This may be obtained using the $O(\mathrm{Re})$ correction to the Stokes velocity field in the inner region ($r \ll \mathrm{Re}^{-1/2}$), derivable by a regular perturbation procedure [10].

Thus, for $\mathrm{Re} \ll 1$, the velocity field close to the sphere may be written as $\boldsymbol{u} = \boldsymbol{u}^{(0)} + \mathrm{Re}\boldsymbol{u}^{(1)} + o(\mathrm{Re})$. In a spherical coordinate system with its polar axis along the vorticity direction, one has for the Stokes velocity field $\boldsymbol{u}^{(0)}$, $\int_0^{2\pi} u_r^{(0)} d\phi = \int_0^{2\pi} u_\theta^{(0)} d\phi = 0$, implying that the near-field streamlines are closed in the absence of inertia. Using Peery's results [10] to determine the components of $\boldsymbol{u}^{(1)}$ for $r \rightarrow 1$, the approximate form of (4) near the sphere is

$$(r-1)^2\{f_0(\theta,\phi) + \mathrm{Re} f_1(\theta,\phi)\}\frac{\partial \Theta}{\partial r} + (r-1) \times \{g_0(\theta,\phi) + \mathrm{Re} g_1(\theta,\phi)\}\frac{\partial \Theta}{\partial \theta} - \Omega_s \frac{\partial \Theta}{\partial \phi} = \frac{1}{\mathrm{Pe}}\nabla^2\Theta, \tag{9}$$

for small Re, where $\int_0^{2\pi} f_0 d\phi = \int_0^{2\pi} g_0 d\phi = 0$, and

$$\int_0^{2\pi} f_1(\theta,\phi) d\phi = \left[-\frac{75}{256}\sin^4\theta + \frac{25}{32}\sin^2\theta - \frac{35}{96}\right], \tag{10}$$

$$\int_0^{2\pi} g_1(\theta,\phi) d\phi = \frac{\sin\theta\cos\theta}{384}[140 - 45\sin^2\theta]. \tag{11}$$

The dominant motion close to the sphere is that of a solid-body rotation with angular velocity $-\Omega_s \mathbf{1}_z$. The no-slip boundary condition on the sphere surface ensures that the meridional and radial velocity components in (9) are small, being linear and quadratic functions, respectively, of the radial distance away from the sphere. Averaging (9) over the azimuthal coordinate $\phi \equiv (0, 2\pi)$, one obtains

$$\mathrm{Re}\left[\frac{(r-1)^2}{2\pi}\int_0^{2\pi} f_1 d\phi \frac{\partial \Theta^{(0)}}{\partial r} + \frac{(r-1)}{2\pi}\int_0^{2\pi} g_1 d\phi \frac{\partial \Theta^{(0)}}{\partial \theta}\right] = \frac{1}{\mathrm{Pe}}\nabla^2\Theta^{(0)}, \tag{12}$$

where we have used $\Theta(r,\theta,\phi) \approx \Theta^{(0)}(r,\theta)$, since the rapid convection along the azimuthal coordinate, while not leading to any net transport of heat, renders the temperature gradient in this direction asymptotically small. Evidently, (12) implies that it is the ϕ-averaged $O(\mathrm{Re})$ convection that enhances the heat transfer at large Pe. In particular, with $y = r - 1$, the leading order balance between the convective terms and the radial diffusive term in (12) reduces to $\mathrm{Re}\, y \sim \mathrm{Pe}^{-1}/y^2$, giving $y \sim O(\mathrm{Pe\,Re})^{-1/3}$ for the boundary layer thickness. Thus, using the rescaled boundary layer coordinate, $\eta = (\mathrm{Re\,Pe})^{1/3}(r-1)$, one obtains

$$\eta^2\left(\int_0^{2\pi} f_1 d\phi\right)\frac{\partial \Theta^{(0)}}{\partial \eta} + \eta\left(\int_0^{2\pi} g_1 d\phi\right)\frac{\partial \Theta^{(0)}}{\partial \theta} = \frac{\partial^2 \Theta^{(0)}}{\partial \eta^2}. \tag{13}$$

Hereafter, the analysis proceeds along standard lines [6]. Defining a similarity variable $s = \eta/h(\cos\theta)$, with $h(\cos\theta)$ characterizing the angular dependence of the boundary layer thickness, one finds

$$\Theta^{(0)}(s) = \frac{1}{\Gamma(\frac{4}{3})} \int_s^\infty e^{-s'^3} ds', \qquad (14)$$

where

$$h(\cos\theta) = \frac{9^{1/3}}{\sin^{1/2}\theta\left(\int_0^{2\pi} g_1 d\phi\right)^{1/2}} \times \left[\int_0^\theta d\theta' \sin^{3/2}\theta' \left(\int_0^{2\pi} g_1 d\phi\right)^{1/2}\right]^{1/3}. \qquad (15)$$

The resulting dimensionless heat flux is found to be [13]

$$\mathrm{Nu} = 0.33(\mathrm{Re\,Pe})^{1/3} + O(1), \qquad (16)$$

where the $O(1)$ correction arises both from terms neglected in the boundary layer approximation above, and from the thermal wake at $\theta = \frac{\pi}{2}$. Even for a modest particle Reynolds number of about 0.3, Nu given by (16) exceeds its inertialess diffusion-limited value when $\mathrm{Pe} > 8800$.

The above bifurcation phenomenon must, in fact, occur for a neutrally buoyant torque-free sphere in any planar shearing flow provided that the ambient velocity gradient at its location is nonzero. The velocity disturbance due to the sphere being asymptotically small near its surface, the flow in this region approximates a solid-body rotation. The reversibility of the Stokes equations then implies the existence of closed streamlines in the vicinity of the rotating sphere at $\mathrm{Re} = 0$. Finite Re should again lead to a convective enhancement, although the detailed streamline topology depends on the nature of the ambient flow. A calculation of the Nusselt number for the case of a torque-free sphere in a planar linear flow [14], together with a demonstration of the symmetry-breaking bifurcation for a nonlinear flow, is presented elsewhere [15].

Microscale inertia will also affect transport processes in particle-laden turbulent flows. On the scale of sub-Kolmogorov particles, turbulence may be regarded as a stochastic linear flow. Based on the degenerate character of the inertialess flow field around a particle in any planar linear flow, Batchelor [16] inferred that the convective heat transfer from such particles only depends on the rate of extension in the vorticity direction. At finite Re, this simplistic scenario is no longer true; the extensional components in the plane transverse to the vorticity vector will, for instance, induce an $O(\mathrm{Re})$ flux that acts to retard the inertialess convective flux [17].

The role of inertia in convective enhancement discussed here has apparently been overlooked despite the solution for the velocity field at small Re, for distances smaller than the inertial screening length, being available since Peery [10]. It appears that the earlier efforts of Acrivos and coworkers on heat transfer from a torque-free cylinder in simple shear, wherein inertia does not effect any qualitative change, led to the erroneous belief, most recently evidenced in the work of Mikulencak and Morris [18], that a similar situation prevails in three dimensions. The dimensionality of space proves crucial in this context, however. A two-dimensional phase plane resulting from a solenoidal velocity field can only have centers and saddles as fixed points; physically, the pressure field adjusts itself to yield closed albeit asymmetric orbits at any finite Re [3,19]. This is no longer true in three dimensions. Owing to the additional degree of freedom, viz motion in the z direction, the $O(\mathrm{Re})$ pressure field does not completely offset the centrifugal force field, and the resulting imbalance alters the streamline topology.

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