Thermal Spectral Functions of Strongly Coupled $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory

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We use the gauge-gravity duality conjecture to compute spectral functions of the stress-energy tensor in finite-temperature $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in the limit of large N_c and large 't Hooft coupling. The spectral functions exhibit peaks characteristic of hydrodynamic modes at small frequency, and oscillations at intermediate frequency. The nonperturbative spectral functions differ qualitatively from those obtained in perturbation theory. The results may prove useful for lattice studies of transport processes in thermal gauge theories.

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Introduction.—Theoretical analysis of the properties of strongly interacting hot and dense matter is a hard problem. Even when the density of baryons in thermal equilibrium is negligible, perturbative QCD calculations are only reliable at temperatures which are much higher than the temperature of the deconfinement transition [1]. While lattice simulations can provide an account of equilibrium thermodynamic properties of the theory, questions involving real-time dynamics such as momentum transport, thermalization, and various production rates are much harder to answer. For near-equilibrium states, physical information is contained in equilibrium response functions: for example, the dilepton rate is proportional to the spectral function of vector currents, and the viscosities are determined by the spectral function of the relevant components of the stress-energy tensor.

Thus it is valuable to study models where analytic results for the real-time response functions can be derived. A number of recent studies focused on a particular model, the $\mathcal{N} = 4$ supersymmetric $SU(N_c)$ Yang-Mills (SYM) theory at finite temperature. The interest in this theory is due to Maldacena's gauge-string duality conjecture [2] which provides an effective description of the theory's nonperturbative regime in terms of semiclassical gravity in a five-dimensional asymptotically anti-de Sitter (AdS) space.

The SYM theory is conformal, and has only one tunable parameter, the 't Hooft coupling λ. Thermal equilibrium is characterized by the blackbody equation of state at any nonzero temperature, and cannot (at large 't Hooft coupling) be viewed as a gas of weakly interacting quasiparticles. The standard hydrodynamic singularities in two-point functions of conserved currents in SYM theory were found from the dual gravity description in the limit of large λ and large N_c [3,4]. The ratio of the shear viscosity to volume entropy density in this theory has been found to be $\eta/s = 1/4\pi$ [5,6], which is a much smaller number than the corresponding result in a weakly coupled gauge theory.

In this Letter, we focus on the full spectral functions (rather than their small-frequency limit) of the stress-energy tensor in the SYM theory at large 't Hooft coupling and large N_c. The knowledge of the full spectral function is important for two reasons. On the one hand, our results provide the first example of a nonperturbative spectral function calculation in a strongly coupled four-dimensional gauge theory at finite temperature, obtained without performing lattice simulations. On the other hand, the nonperturbative spectral functions of SYM theory may be useful for lattice computations of transport coefficients in realistic gauge theories at temperatures not too far from the deconfinement transition. Indeed, a Euclidean correlation function (computed on the lattice) is proportional to the integral of the real-time spectral function over all frequencies. In the simplest method of reconstructing the spectral function from the Euclidean data, one first assumes an ansatz for the spectral function, and then fits the parameters of the ansatz to the lattice data [7]. This method has been applied to the computation of the shear viscosity in QCD [8]. Theoretical understanding of what the correct nonperturbative ansatz might be is of primary importance for this approach. Our results for the nonperturbative spectral functions of SYM theory may therefore prove useful for lattice studies of transport processes in thermal gauge theories.

Correlation functions from gauge-gravity duality.—The spectral function $\chi_{\mu\nu,\alpha\beta}(k)$ is defined as

$$\chi_{\mu\nu,\alpha\beta}(k) = \int d^4x e^{-ikx} \langle [T_{\mu\nu}(x), T_{\alpha\beta}(0)] \rangle. \tag{1}$$

It is proportional to the imaginary part of the retarded Green's function, $\chi_{\mu\nu,\alpha\beta}(k) = -2\,\mathrm{Im} G_{\mu\nu,\alpha\beta}(k)$, where

$$G_{\mu\nu,\alpha\beta}(k) = -i \int d^4x e^{-ikx} \theta(x^0) \langle [T_{\mu\nu}(x), T_{\alpha\beta}(0)] \rangle, \tag{2}$$

and is an odd, real function of k. The shear and bulk viscosities are proportional to the zero-frequency slope of the specific components of the spectral function, for example,

$$\eta = \lim_{k^0 \to 0} \frac{1}{2k^0} \chi_{xy,xy}(k^0, \boldsymbol{k} = 0). \tag{3}$$

The retarded correlation function of the stress-energy tensor of SYM theory admits a simple decomposition (we follow the notational conventions of Ref. [9]). At zero temperature, Lorentz symmetry combined with conservation and tracelessness of $T_{\mu\nu}$ implies that $G_{\mu\nu,\alpha\beta}(k)$ has the form

$$G_{\mu\nu,\alpha\beta}(k) = H_{\mu\nu,\alpha\beta} G_S(k^2),$$

where $H_{\mu\nu,\alpha\beta} = \frac{1}{2}(P_{\mu\alpha}P_{\nu\beta} + P_{\mu\beta}P_{\nu\alpha}) - \frac{1}{3}P_{\mu\nu}P_{\alpha\beta}$ is a projector onto conserved traceless symmetric tensors, $P_{\mu\nu} = \eta_{\mu\nu} - k_\mu k_\nu / k^2$, and $k^2 = -(k^0)^2 + \boldsymbol{k}^2$. At nonzero temperature, $G_{\mu\nu,\alpha\beta}$ in a conformal theory can be described by three symmetry channels

$$G_{\mu\nu,\alpha\beta}(k) = S_{\mu\nu,\alpha\beta}G_1 + Q_{\mu\nu,\alpha\beta}G_2 + L_{\mu\nu,\alpha\beta}G_3, \tag{4}$$

where $S_{\mu\nu,\alpha\beta}$, $Q_{\mu\nu,\alpha\beta}$, and $L_{\mu\nu,\alpha\beta}$ are the appropriate orthogonal projectors which provide three independent Lorentz index structures [9]. Choosing the spatial momentum along x^3, $k_\mu = (-\omega, 0, 0, q)$, the components of the correlation function are

$$G_{tx^1,tx^1}(k) = \frac{1}{2} \frac{q^2}{\omega^2 - q^2} G_1(\omega, q), \tag{5a}$$

$$G_{tt,tt}(k) = \frac{2}{3} \frac{q^4}{(\omega^2 - q^2)^2} G_2(\omega, q), \tag{5b}$$

$$G_{x^1x^2,x^1x^2}(k) = \frac{1}{2} G_3(\omega, q), \tag{5c}$$

with all other related to the above by the rotation invariance. As a function of complex ω, in the low-frequency limit, $G_1(\omega, q)$ has a shear-mode singularity, $G_2(\omega, q)$ has a sound-mode singularity, and $G_3(\omega, q)$ has no hydrodynamic singularities. In the limit of vanishing 3-momentum, $G_1(\omega) = G_2(\omega) = G_3(\omega)$. At zero temperature, $G_1 = G_2 = G_3 = G_S$.

In the regime of large 't Hooft coupling, the three scalar functions $G_a(\omega, q)$ can be computed using the gauge-gravity duality recipe [9,10]. The duality essentially reduces the computation of a two-point correlation function to a boundary-value problem for a linear ordinary differential equation. For the zero-temperature theory, the retarded two-point function can be found, for example, in [10] (at zero temperature, the dual gravity result for the correlator in strongly coupled $\mathcal{N} = 4$ SYM theory coincides with the one obtained in free field theory, due to a nonrenormalization theorem [11]),

$$G_S(k) = \frac{N_c^2 k^4}{32\pi^2}[\ln|k^2| - i\pi\theta(-k^2)\,\mathrm{sign}\omega]. \tag{6}$$

In order to compute the retarded correlators at nonzero temperature, one has to analyze the "wave equations" (one for each symmetry channel) which describe propagation of the corresponding metric perturbations in the AdS-Schwarzschild background spacetime of the dual description. The differential equations are of the form

$$\frac{d^2}{du^2}Z_a(u) + p_a(u)\frac{d}{du}Z_a(u) + q_a(u)Z_a(u) = 0, \tag{7}$$

where the coefficients $p_a(u)$, $q_a(u)$ [to be specified shortly] depend on the dimensionless frequency $\mathfrak{w} \equiv \omega/2\pi T$ and momentum $\mathfrak{q} \equiv q/2\pi T$, and $a = 1, 2, 3$ labels the three symmetry channels. The coordinate u ranges from 0 to 1, where $u = 0$ corresponds to the boundary of the asymptotically AdS spacetime, and $u = 1$ corresponds to the event horizon of the background.

For all three Eqs. (7), the characteristic exponents at $u = 0$ are equal to 0 and 2, and the exponents at $u = 1$ are $\pm i\mathfrak{w}/2$. Information about the retarded correlation function is encoded in the solutions to Eq. (7) which satisfy the incoming-wave condition at the horizon, corresponding to the exponent $-i\mathfrak{w}/2$ at $u = 1$. The correct solution is thus of the form $Z_a(u) = (1-u)^{-i\mathfrak{w}/2}F_a(u)$, where $F_a(u)$ is a regular function at the horizon. The solution satisfying the incoming-wave condition at the horizon can be written as a linear combination of two independent local solutions at $u = 0$,

$$Z_a(u) = \mathcal{A}_a Z_a^{\mathrm{I}}(u) + \mathcal{B}_a Z_a^{\mathrm{II}}(u), \tag{8}$$

where $Z_a^{\mathrm{I}}(u)$ and $Z_a^{\mathrm{II}}(u)$ are given by their standard Frobenius expansions [12] as

$$Z_a^{\mathrm{I}} = 1 + b_{a\mathrm{I}}^{(1)}u + h_a Z_a^{\mathrm{II}}(u)\ln u + b_{a\mathrm{I}}^{(2)}u^2 + \cdots, \tag{9}$$

$$Z_a^{\mathrm{II}} = u^2(1 + b_{a\mathrm{II}}^{(1)}u + b_{a\mathrm{II}}^{(2)}u^2 + \cdots). \tag{10}$$

All the coefficients $b_{a\mathrm{I,II}}^{(j)}$ (except $b_{a\mathrm{I}}^{(2)}$) and h_a are determined by the recursion relations obtained by substituting the above expansion in the differential Eq. (7). The coefficient $b_{a\mathrm{I}}^{(2)}$ is left as a free parameter, reflecting the fact that one can always redefine the local solutions by adding a constant multiple of $Z_a^{\mathrm{II}}(u)$ to $Z_a^{\mathrm{I}}(u)$. Without loss of generality, we set $b_{a\mathrm{I}}^{(2)} = 0$ thus fixing the definition of $Z_a^{\mathrm{I,II}}(u)$. The retarded functions G_a are then given by [13],

$$G_a(\omega, q) = -\pi^2 N_c^2 T^4 \frac{\mathcal{B}_a(\omega, q)}{\mathcal{A}_a(\omega, q)}. \tag{11}$$

As is evident from Eqs. (9) and (10), the coefficient $\mathcal{A}$ is given by the boundary value of the solution, $\mathcal{A}_a = \lim_{u\to 0} Z_a(u)$, while the coefficient $\mathcal{B}$ can be expressed in terms of the boundary value of the second derivative of the solution,

$$\mathcal{B}_a = \frac{1}{2}\lim_{u\to 0}[Z_a''(u) - 2\mathcal{A}_a h_a \ln(u)] - \frac{3}{2}\mathcal{A}_a h_a. \quad (12)$$

From the recursion relations one determines $h_a = -\frac{1}{2} \times (\mathfrak{q}^2 - \mathfrak{w}^2)^2$, which is an analytic function of $\mathfrak{w}$ and $\mathfrak{q}$, and therefore represents a contact term which we drop. The retarded correlator is therefore equal to

$$G_a = -\pi^2 N_c^2 T^4 \lim_{u\to 0}\left(\frac{Z_a''(u)}{2Z_a(u)} - h_a \ln(u)\right). \quad (13)$$

Spectral functions of the stress-energy tensor.—In order to determine the spectral function for transverse stress, we need to solve the master Eq. (7) whose coefficients are given by [10]

$$p_3(u) = -\frac{1+u^2}{uf}, \qquad q_3(u) = \frac{\mathfrak{w}^2 - \mathfrak{q}^2 f}{uf^2}, \quad (14)$$

where $f = 1 - u^2$. Solving Eq. (7) numerically, we find $\chi_{xy,xy}(\omega, q)$ as explained in the previous section. The spectral function is linear in $\mathfrak{w}$ for small frequencies, $\chi_{xy,xy} = (\mathfrak{w}/2)\pi^2 N_c^2 T^4[1 + O(\mathfrak{w}^2, \mathfrak{q}^2)]$, then increases monotonically, and at large frequencies it asymptotes to the zero-temperature result $\chi_{xy,xy}^{T=0} = \frac{\pi}{2}(\mathfrak{q}^2 - \mathfrak{w}^2)^2 \pi^2 N_c^2 T^4 \theta(-k^2)$. When this zero-temperature contribution is subtracted, the resulting function exhibits oscillations which damp rapidly as frequency grows. The oscillations appear around $\omega = q$, and their amplitude grows with q. The unsubtracted $\chi_{xy,xy}$ is positive, as it should be. Figure 1 shows graphs of $\chi_{xy,xy}$ for several values of three-momentum. The numerical procedure outlined above allows one to compute both real and imaginary parts of $G_{\mu\nu,\alpha\beta}$. The real part which can in principle be reconstructed from the spectral function, also exhibits oscillations which damp as frequency grows [14]. The shear viscosity follows from the Kubo formula (3), $\eta = \pi N_c^2 T^3/8$, and is in agreement with the earlier results [5,6].

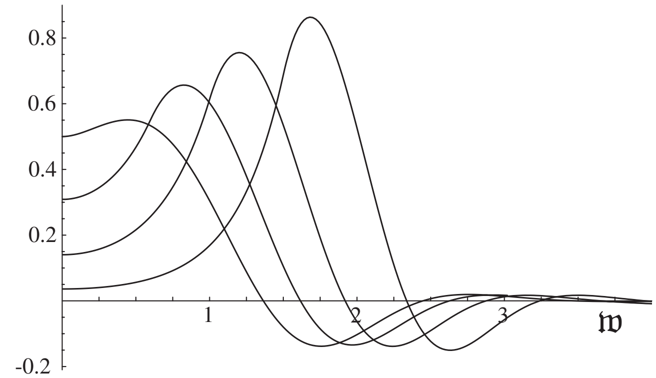

FIG. 1. The finite-temperature part of the spectral function for transverse stress $(\chi_{xy,xy} - \chi_{xy,xy}^{T=0})/\mathfrak{w}$, plotted in units of $\pi^2 N_c^2 T^4$ as a function of dimensionless frequency $\mathfrak{w} \equiv \omega/2\pi T$. Different curves correspond to values of the dimensionless spatial momentum $\mathfrak{q} \equiv q/2\pi T$ equal to 0, 0.6, 1.0, and 1.5.

In the shear channel, the coefficients of the master Eq. (7) are given by [9]

$$p_1(u) = \frac{(\mathfrak{w}^2 - \mathfrak{q}^2 f)f + 2u^2\mathfrak{w}^2}{uf(\mathfrak{q}^2 f - \mathfrak{w}^2)}, \qquad q_1(u) = \frac{\mathfrak{w}^2 - \mathfrak{q}^2 f}{uf^2}. \quad (15)$$

The spectral function $\chi_{tx,tx}(\omega, q)$ is shown in Fig. 2. At small momentum, the spectral function exhibits a narrow peak at $\mathfrak{w} = \mathfrak{q}^2/2$, characteristic of the hydrodynamic shear mode.

In the sound channel, the coefficients of the master Eq. (7) are given by [9]

$$p_2(u) = -\frac{3\mathfrak{w}^2(1+u^2) + \mathfrak{q}^2(2u^2 - 3u^4 - 3)}{uf[3\mathfrak{w}^2 + \mathfrak{q}^2(u^2 - 3)]}, \quad (16)$$

$$q_2(u) = \frac{3\mathfrak{w}^4 + \mathfrak{q}^4(3 - 4u^2 + u^4) + \mathfrak{q}^2(4u^2\mathfrak{w}^2 - 6\mathfrak{w}^2 - 4u^3 f)}{uf^2[3\mathfrak{w}^2 + \mathfrak{q}^2(u^2 - 3)]}. \quad (17)$$

The spectral function $\chi_{tt,tt}(\omega, q)$ is shown in Fig. 2. At small momenta, the spectral function exhibits a narrow peak at $\mathfrak{w} = \mathfrak{q}/\sqrt{3}$, characteristic of the hydrodynamic sound mode. For both $\chi_{tx,tx}$ and $\chi_{tt,tt}$, the finite-temperature contributions have oscillatory behavior, similar to the one seen in $\chi_{xy,xy}$.

Discussion.—It is easy to understand the limiting behavior of the spectral functions found above. The ω, q dependence at small frequency is predicted by the linearized hydrodynamics, while the large ω dependence is fixed by the scale invariance of the SYM theory. An intriguing feature is the presence of oscillations in the finite-temperature contribution, which appear around $\omega = q$, and then decay rapidly. Mathematically, such damped oscillations are due to the characteristic asymptotic behavior, $\sim \exp(-\alpha\mathfrak{w})$, of the solutions to Eq. (7), where α is a complex number [10,15]. They reflect the presence of an infinite sequence of poles in the lower half-plane of the retarded correlators [9].

One can compare the spectral functions of the strongly coupled SYM theory with the perturbative results in a weakly coupled scalar or pure gauge theory discussed in Ref. [16]. At weak coupling, the spectral function for transverse stress at $q = 0$ grows linearly at small frequency, and is proportional to ω^4 at asymptotically high frequency. In between, however, there is a range where the

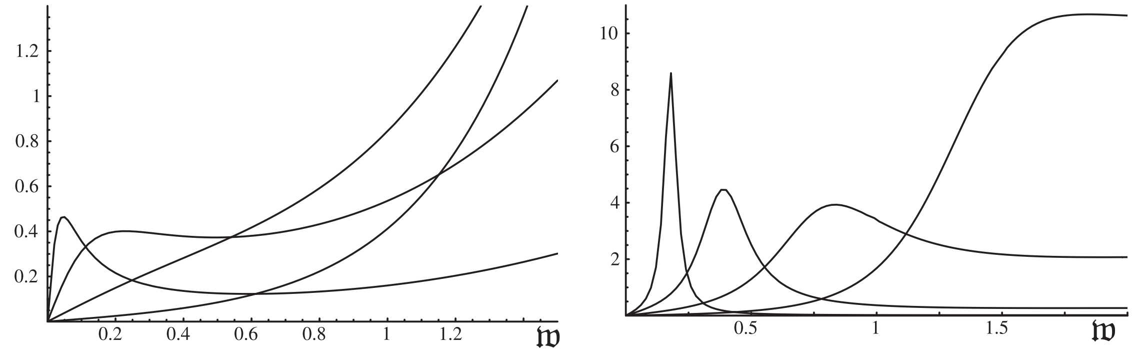

FIG. 2. Left: spectral function for transverse momentum density, $\chi_{tx,tx}$, plotted in units of $\pi^2 N_c^2 T^4$, as a function of dimensionless frequency $\mathfrak{w} \equiv \omega/2\pi T$. Different curves correspond to values of the dimensionless spatial momentum $\mathfrak{q} \equiv q/2\pi T$ equal to 0.3, 0.6, 1.0, and 1.5. At large $\mathfrak{w}$, the curves asymptote to the zero-temperature result $\frac{\pi}{2}\mathfrak{q}^2(\mathfrak{w}^2 - \mathfrak{q}^2)$. Right: spectral function for energy density, $\chi_{tt,tt}$, plotted in units of $\pi^2 N_c^2 T^4$, as a function of dimensionless frequency $\mathfrak{w} \equiv \omega/2\pi T$. Different curves correspond to values of the dimensionless spatial momentum $\mathfrak{q} \equiv q/2\pi T$ equal to 0.3, 0.6, 1.0, and 1.5. At large $\mathfrak{w}$, the curves asymptote to the zero-temperature result $\frac{\pi}{2} 4\mathfrak{q}^4/3$.

spectral function decreases as $1/\omega$. This behavior at intermediate frequencies at weak coupling differs from what one observes in the strongly coupled SYM theory, where the spectral function grows monotonically for all ω. Based on the numerical results for $\chi_{\mu\nu,\alpha\beta}(\omega, q)$, one may propose the following ansatz for the zero-momemtum spectral function for transverse stress

$$\chi_{xy,xy}(\omega) = \chi_{xy,xy}^{T=0}(\omega) + \mathrm{Im}\sum_i c_i e^{-\alpha_i \omega}, \qquad (18)$$

where c_i, are real coefficients, and $\mathrm{Re}(\alpha_i) > 0$ (assuming positive frequency).

In this Letter, we computed the spectral functions in the simplest thermal gauge theory with a known gravity dual. It should also be feasible to find spectral functions for a class of nonconformal gauge theories, such as the one analyzed in Ref. [17].

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Note added.—Recently, we became aware of work on the same subject by D. Teaney [18].

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