Adiabatic Swimming in an Ideal Quantum Gas

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(Received 31 August 2005; published 6 April 2006)

Interference effects are important for swimming of mesoscopic systems that are small relative to the coherence length of the surrounding quantum medium. Swimming is geometric for slow swimmers and the distance covered in each stroke is determined, explicitly, in terms of the on-shell scattering matrix. Remarkably, for a one-dimensional Fermi gas at zero temperature we find that slow swimming is topological: the swimming distance covered in one stroke is quantized in half integer multiples of the Fermi wavelength. In addition, a careful choice of the swimming stroke can eliminate dissipation.

DOI: 10.1103/PhysRevLett.96.130602 PACS numbers: 05.70.Ln, 02.40.−k, 05.40.−a, 73.23.−b

The theory of classical swimming studies how a cyclic change in the shape of a swimmer immersed in a fluid leads to a change in its location. The theory is both elegant and practical [1,2] and has been applied to the swimming and flying of organisms, robots [3], and microbots [4,5].

A classical medium may be viewed as a limiting case of quantum medium when the quantum coherence length is small compared to the size of the swimmer and interference is negligible. Quantum mechanics takes over once the swimmer is small enough. When this is the case, interference effects may significantly affect the swimming. We shall focus on the case of slow (adiabatic) swimmers, where the theory turns out to be geometric. In the theory the swimmer is viewed as a (time-dependent) scatterer and the main result is a formula for the swimming distance in one swimming stroke in terms of the on-shell scattering matrix. This point of view also gives a different perspective on the issue of quantum friction [6,7].

It may be instructive to explain the difference between geometrical and nongeometrical modes of swimming. In general, nonadiabatic swimmers move by transferring momentum to the ambient medium. For example, a swimmer may emit photons, or electron-hole pairs that carry momentum and accelerate the swimmer. Using such a mechanism it is possible to swim in vacuum, e.g., using the dynamic Casimir effect [8]. This mode of locomotion is effective for swift swimmers. Adiabatic or geometric swimmers, in contrast, swim without transferring momentum to the ambient medium [1] (this shall be further discussed below). In particular, the swimmers that we consider do not swim by emitting quanta (e.g., photons) that propel them.

We focus on swimming in a one-dimensional ideal Fermi gas at low temperatures where interference effects are especially strong. This situation leads to a remarkable quantization of the swimming distance. At the same time, our methods are more general and can be also applied to Bose gas and finite temperature and also to swimming in three dimensions.

Let us now turn to a more precise description of the setting. The swimmer, a q swimmer, is an object with internal degrees of freedom which, we assume, are slow and classical while the degrees of freedom of the medium are fast and quantum. Swimming is accomplished by the internal degrees of freedom, the controls, undergoing periodic cycles. An example of a q swimmer might be a molecule immersed in a quantum gas. The control of the internal configuration might be either external or internal (due to autonomous dynamics). As we shall see, there is as intimate connection between quantum pumps [9] and q swimming.

Consider a simple model of a swimmer made of n disconnected spheres of radii a_j immersed in either a classical or quantum medium. The swimmer can control the $n-1$ relative distances $\ell_i = X_i - X_n$, $i = 1, \ldots, n-1$ between the centers X_i of the spheres and the radii a_j. Allow the swimmer to change adiabatically the control parameters a_j and ℓ_i. The notion of adiabaticity means that the velocities, e.g., $\dot{X}_j$, $\dot{a}_j$, are small compared with the characteristic velocities of the particles in the medium. By linear response, the force on the jth sphere is given by

$$f_j = -\sum_k \eta_{jk}\dot{X}_k - \sum_k \nu_{jk}\dot{a}_k + \sum_k F_{jk}. \tag{1}$$

Here $F_{ji} = -F_{ij}$ are internal forces acting between i and j spheres and η_{jk}, ν_{jk} are coefficients, which, *a priori*, depend on the state of the swimmer (i.e., the relative distances ℓ_i and the radii of the spheres a_j) and the nature of the medium.

To derive an explicit form of the swimming equation one first needs to make a choice how to designate the position of the swimmer, X. For a swimmer made of n disconnected pieces it is convenient to pick $X = X_n$, the coordinate of one of the components. The total force $\sum_j f_j$ acting on an

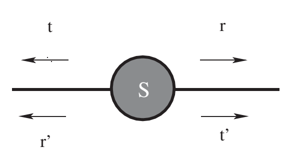

FIG. 1. A scatterer with two scattering channels: (r, t) and (r', t') are the reflection and transmission amplitudes.

adiabatic swimmer must vanish. (This follows from the fact that the friction forces are of first order in the adiabaticity while the acceleration is second order.) This constraint determines the swimming equation

$$-\eta \,\text{đ}X = \sum_{i=1}^{n-1} \eta_i d\ell_i + \sum_{j=1}^{n} \nu_j da_j, \tag{2}$$

where $\eta_k = \sum_{j=1}^n \eta_{jk}$, $\eta = \sum_{j=1}^n \eta_j$, $\nu_k = \sum_{j=1}^n \nu_{jk}$. Swimming is manifestly geometric being independent of the (time) parametrization of the swimming stroke. The notation $\text{đ}X$ stresses that the position of the swimmer will, in general, not integrate to a function on the space of controls: X will not return to its original values when the controls undergo a cycle.

Equation (2) does not determine the coefficients η_i, ν_j, and η. In this sense, the swimming equation, although general, is incomplete. We start by deriving a new formula for the quantum friction η [10]. η expresses the friction acting on an idle q swimmer while it is being dragged at small velocity through the ambient quantum medium. The formula is in the spirit of Landauer formula [11,12]; it is expressed in terms of the scattering data. As usual in the Landauer setting we assume a one-dimensional system. For the sake of simplicity, we focus on the two channel case (see Fig. 1).

The most general on-shell scattering matrix of a one-dimensional time-reversal invariant scatterer anchored at the origin can be parametrized as follows [13]:

$$S_0 \equiv \begin{pmatrix} t & r' \\ r & t' \end{pmatrix} = e^{i\gamma} \begin{pmatrix} i\sin\theta & e^{-i\alpha}\cos\theta \\ e^{i\alpha}\cos\theta & i\sin\theta \end{pmatrix}. \tag{3}$$

The reflection r, r' and transmission t, t' amplitudes are functions of three independent real parameters: α, γ, θ.

The scattering matrix of a scatterer located at X is related to the scattering matrix located at the origin, S_0, by:

$$S(X) = e^{iPX/\hbar} S_0 e^{-iPX/\hbar}, \tag{4}$$

where the on-shell momentum matrix P is defined by

$$P = p(E) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5}$$

$p(E) = \sqrt{2mE}$ for electron gas and $p(E) = E/c$ for photons.

A dragged scatterer may be thought of as a pump. If the velocity of dragging is small compared with the characteristic velocity of the scattered particles, the theory of adiabatic pumps can be applied. In particular, the rate of momentum transfer to the ambient medium is [14]:

$$\dot{\mathcal{P}}(t) = -\frac{1}{2\pi\hbar} \int dE \rho'(E) \,\mathrm{Tr}_E[\mathcal{E}(E,t)P], \tag{6}$$

$\mathcal{E}(E,t) = i\hbar \dot{S} S^*$ is called the energy shift matrix [15]. It depends on a frozen on-shell scattering matrix and its time derivative. Here and after S^* denotes the Hermitian conjugate of S, Tr_E denotes the trace on the scattering channels at fixed energy E, and $\rho(E)$ gives the occupation of the (scattering) states at energy E. If the ambient quantum gas is at thermal equilibrium, $\rho(E) = (e^{\beta(E-\mu)} \pm 1)^{-1}$ is the Fermi-Dirac or Bose-Einstein distribution.

In dragging a scatterer, the time dependence of S comes solely from the change of position, X. From Eq. (4) $\mathcal{E} = \dot{X}[P, S]S^*$. Equation (6) determines the force f on the swimmer: $f = \dot{\mathcal{P}}$.

Define, as usual, the friction coefficient, η, by $f = -\eta \dot{X}$. Combined with Eq. (6) we get a Landauer type formula for the quantum friction

$$\eta = -\frac{1}{4\pi\hbar} \int dE \rho'(E) \,\mathrm{Tr}_E([P, S][P, S]^*). \tag{7}$$

At thermal equilibrium $\rho(E)$ is a decreasing function of the energy. This implies that η is non-negative.

For a Fermi gas at zero temperature $\rho'(E) = -\delta(E - E_F)$. In the two channels case one then has $\eta_F = \frac{2}{\pi\hbar} p^2(E_F)|r(E_F)|^2$. The friction depends only on the momentum and reflection at the Fermi energy, as one expects. Transparent objects are frictionless.

To derive a q swimming equation and fix the coefficients of Eq. (2) we make use of the elementary observation that swimming is dual to pumping. A turning screw can be used to either pump or swim. The difference lies in the setup: in a pump the external forces and torques adjust to satisfy the constraints that the position and orientation of the pump are fixed while in a swimmer the position and orientation of the swimmer adjust to satisfy the constraint that there are no external forces and torques.

Assume that no external forces are applied on a swimmer which can control its scattering matrix. The rate of momentum transfer is still given by Eq. (6). Now, however, the energy shift has two terms: $\mathcal{E} = \mathcal{E}_X + \mathcal{E}_0$. The first $\mathcal{E}_X$ [given in Eq. (6)] arises from the swimmer's change of location, while the second $\mathcal{E}_0 = i\hbar \dot{S}_0 S_0^*$ comes from the swimming stroke.

The total force acting on an adiabatic swimmer must vanish (to first order). This means that $\dot{\mathcal{P}}$ vanishes and the equation of motion for q swimmers [16] is:

$$\eta \dot{X} = \frac{1}{2\pi\hbar} \int dE \rho'(E) \,\mathrm{Tr}_E(\mathcal{E}_0 P), \tag{8}$$

where η is the friction coefficient, given in Eq. (7) and P is given in Eq. (5). In the two channels case the trace on the right-hand side is given by

$$\mathrm{Tr}_E(\mathcal{E}_0 P) = \hbar p(E) |r|^2 \mathrm{Im}[d_t \log(r/r')]. \tag{9}$$

We shall now apply the q swimming equation, Eq. (8), to swimming in a Fermi gas at zero temperature in one dimension. This case is both simple and remarkable for, as we shall see, q swimming turns out to be topological. A small deformation of the swimming stroke does not affect the swimming distance which is an integer multiple of the

Fermi wavelength. This follows immediately from Eqs. (8) and (9) which combine to give:

$$dX = \frac{\lambda_F}{8\pi}\,\mathrm{Im}\,[d\log(r/r')] = \frac{\lambda_F}{4\pi}d\alpha, \tag{10}$$

where λ_F is the Fermi wavelength. The fact that right-hand side is an exact differential of the parameter α has two consequences: first, to swim one must encircle the point where the scatterer is transparent, $r = r' = 0$, and second, the distance covered in a stroke is quantized as a multiple of $\lambda_F/2$. The result is general and does not depend on the specifics of the swimmers.

A swimmer will normally not have direct control on parameters α. Rather, it will control some physical parameters that will determine the scattering matrix (see examples below). Consider a swimmer with two independent controls. A stroke is a closed path in the plane of controls [17]. Since the reflection r is a complex valued function of the controls, one expects that by adjusting two controls, one can find points where $r = 0$. With each such point of transparency one can associate an (integer) index that counts how many times r rotates around the origin in one cycle around the point. We call the index the *vorticity*. The swimming distance in a closed stroke is proportional to the vorticity enclosed by the path.

Let us now consider two examples of topological swimmers.

Pushmepullyou.—Consider "a molecule" made of two scatterers. The scattering matrices associated with each scatterer have $\alpha_j = \gamma_j = 0$ for both scatterers and $r_1 = \cos\theta_1$, $r_2 = \cos\theta_2$. The two scatterers are separated by distance ℓ (see Fig. 2). The swimmer can control ℓ, and the ratio r_1/r_2. The total scattering matrix of the swimmer can then be computed by considering the multiple scattering processes between the two scatterers. A computation yields for the zeros of total reflection of r the solutions of: $e^{2ik\ell}r_1 - r_2 = 0$. It follows that the vortices occur when $r_1 = \pm r_2$ and the distance 2ℓ is an integer (half and integer) multiple of wavelengths. Since the zeros are simple the vorticities are ± 1.

Three linked spheres.—Consider a swimmer [proposed by Najafi and Golestanian in [4] in a different context], made of three identical scatterers on a line, separated by distances ℓ_1 and ℓ_2 (see Fig. 2). Take $\alpha_j = \gamma_j = 0$ and $r_j = \cos\theta$, independent of $j = 1, 2, 3$. The swimmer, a vibrating "trimer," now controls only the two distances ℓ_1, ℓ_2. The total reflection, r, vanishes provided $e^{i2k\ell_1} + e^{-i2k\ell_2} - \cos^2\theta e^{i2k(\ell_1-\ell_2)} - 1 = 0$. The vortices evidently occur on a pair of lattices in the plane (ℓ_1, ℓ_2) (see Fig. 2). Once again, since the zeros are simple, the vorticities are ± 1.

Equations (8) and (10) define a connection $đX$ in the space of controls. The distance covered by the swimmer in a one stroke $\mathcal{C}$ is then given by $\Delta X = \int_{\mathcal{C}} đX$. When there are only two control parameters (x, y) (as in the examples above), an application of the Stokes formula gives for the displacement $\Delta X = \iint f(x, y)dxdy$, where $f(x, y)$ is the (scalar) *curvature*, and the domain of integration has the boundary $\mathcal{C}$. A plot of the curvature f is often an efficient way to describe a swimmer (see, e.g., Figs. 3 and 4).

It is instructive to see how topological Fermi swimmers are affected by temperature. At temperature T a region proportional to T near the Fermi energy will contribute to the integral in Eq. (8). This makes the curvature a smooth function (rather than a collection of delta functions) on the space of controls. The total curvature enclosed in a path will now depend smoothly on the path. The temperature scale is determined by $T_0 = h^2/4mk_B\lambda_F\ell$ where k_B is Boltzmann constant and m the mass of the scattered particle. At this temperature the support of f near a vortex becomes comparable with the distance between vortices. From the definition of T_0 it follows that the larger the swimmer (the larger ℓ) the more sensitive it is to temperature.

Pushmepullyou and the three linked spheres have an essentially different behavior at high temperatures. Since the neighboring vortexes of three linked spheres (at $T = 0$) are of opposite sign, the smearing at high temperatures leads to vanishing curvature at high temperatures (see

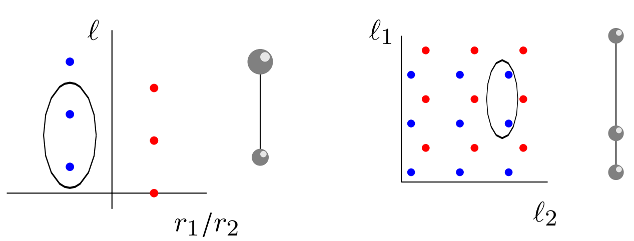

FIG. 2 (color online). The vortex structure, at $T = 0$, of Pushmepullyou (left) and the three linked spheres (right). The swimming, in both cases, is topological: the distance covered in one stroke is proportional to the sum of the vorticities encircled by the path (the ellipses in the figures). The red (light gray) dots have vorticity $+1$ and the blue (dark gray) dots have vorticity -1.

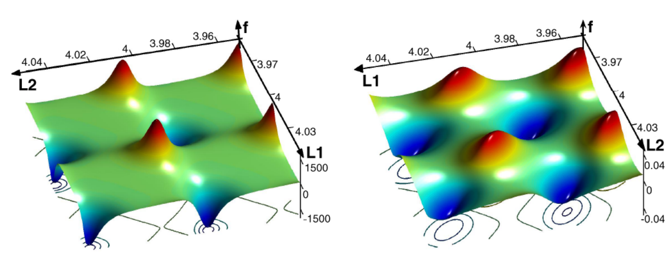

FIG. 3 (color online). The curvature for the three linked spheres, shown in a 3D plot as a function of control parameters at low temperature, $T = T_0/4$ (left) and high temperature, $T = 4T_0$ (right). The height of the peaks in the figure on the left is about a million times the height of the peaks on the right. The distance covered in one stroke is the total curvature enclosed by the stroke.

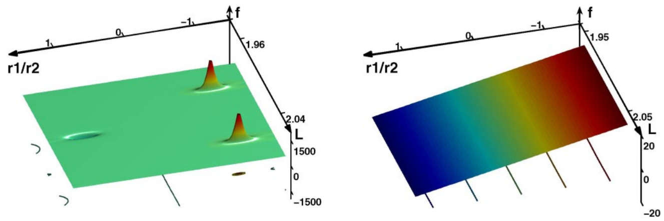

FIG. 4 (color online). The curvature at finite temperatures for Pushmepullyou, shown in a 3D plot as a function of control parameters for low temperatures (left) and high temperatures (right). In contrasts with the three linked spheres, Pushmepullyou has finite curvature also at high temperatures.

Fig. 3). Thus the three linked spheres do not swim effectively at high temperatures. On the other hand, the smearing of the vortices of Pushmepullyou (at $T = 0$) does not lead to mutual cancellation (see Fig. 4). That means Pushmepullyou can swim effectively also at high temperatures.

In the course of its motion a swimmer will, in general, transfer energy to the medium and dissipate energy. As we shall now show, in the adiabatic limit swimming without dissipation is possible in one-dimensional Fermi gas at zero temperature.

We define the dissipation, $\dot{D}$, as the leading order (in the adiabaticity parameter) of difference between the outgoing and incoming energy current. From [14]:

$$\dot{D} = \frac{\hbar}{4\pi}\,\mathrm{Tr}_{E_F}(\dot{S}\dot{S}^*) \geq 0. \tag{11}$$

One of the consequences of this relation is that a non-dissipating swimmer is indistinguishable from a static scatterer (since $\dot{S} = 0$). Clearly, there is no dissipation in the former, and there should therefore be no dissipation in the latter.

To see why it is possible to swim without dissipation, write Eq. (11), (in the two channel case), in the form:

$$\dot{D} = \frac{\hbar}{2\pi}[(\dot{\alpha} - 2k_F\dot{X})^2\cos^2\theta + \dot{\theta}^2 + \dot{\gamma}^2]. \tag{12}$$

The first term vanishes by Eq. (10). Thus if $\dot{\theta} = \dot{\gamma} = 0$ there is no dissipation. We call a swimming without dissipation "superswimming." Note that the ambient medium, an ideal gas, has no gap in the spectrum and so, unlike a superfluid, does not offer protection from dissipation [for friction effects in superfluids, see, e.g., [18]]. Indeed, energy would be required to drag the swimmer through the medium. It is only by choosing a swimming stroke carefully that the superswimmer avoids dissipating energy.

A superswimmer needs a larger space of controls to ensure that $\dot{\theta} = \dot{\gamma} = 0$. (For instance, to make a superswimmer out of Pushmepullyou one needs to control also the overall phase $e^{i\gamma}$ of the two individual scatterers.) Superswimmers are, in general, not transparent and one still needs to invest power to drag them. Only when the superswimmer swims on its own, there is no transfer of energy to the ambient medium.

We note that no dissipation, $\dot{S} = 0$, implies Eq. (10) and hence implies quantization.

This work is supported in parts by the EU Grant No. HPRN-CT-2002-00277, the ISF, the Minerva Foundation, and the fund for the promotions of research at the Technion.

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