Detecting Topological Order in a Ground State Wave Function

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(Received 25 October 2005; published 24 March 2006)

A large class of topological orders can be understood and classified using the string-net condensation picture. These topological orders can be characterized by a set of data $(N, d_i, F^{ijk}_{lmn}, \delta_{ijk})$. We describe a way to detect this kind of topological order using only the ground state wave function. The method involves computing a quantity called the "topological entropy" which directly measures the total quantum dimension $D = \sum_i d_i^2$.

DOI: 10.1103/PhysRevLett.96.110405 PACS numbers: 11.15.−q, 03.65.Ud, 11.25.−w, 71.10.−w

Introduction.—Until recently, the only known physical characterizations of topological order [1] involved properties of the Hamiltonian—e.g., quasiparticle statistics [2], ground state degeneracy [3,4], and edge excitations [1]. In this Letter, we demonstrate that topological order is manifest not only in these dynamical properties but also in the basic entanglement of the ground state wave function. We hope that this characterization of topological order can be used as a theoretical tool to classify trial wave functions—such as resonating dimer wave functions [5], Gutzwiller projected states [6], or quantum loop gas wave functions [7]. In addition, it may be useful as a numerical test for topological order. Finally, it demonstrates definitively that topological order is a property of a wave function, not a Hamiltonian.

Main Result.—We focus on the $(2+1)$-dimensional case (though the result can be generalized to any dimension). Let Ψ be an arbitrary wave function for some two-dimensional lattice model. For any subset A of the lattice, one can compute the associated quantum entanglement entropy S_A [8]. The main result of this Letter is that one can determine the "total quantum dimension" D of Ψ by computing the entanglement entropy S_A of particular regions A in the plane. Normal states have $D = 1$ while topologically ordered states have $D > 1$. Thus, this result provides a way to distinguish topologically ordered states from normal states, using only the wave function.

More specifically, consider the four regions A_1, A_2, A_3, A_4 drawn in Fig. 1. Let the corresponding von Neumann entanglement entropies be S_1, S_2, S_3, S_4. Consider the linear combination $(S_1 - S_2) - (S_3 - S_4)$, computed in the limit of large, thick annuli, $R, r \to \infty$. The main result of this Letter is that

$$(S_1 - S_2) - (S_3 - S_4) = -\log(D^2), \tag{1}$$

where D is the total quantum dimension of the topological order associated with Ψ. Here, $D = \sum_i d_i^2$ [9] for a topological order described by a string-net condensate $(N, d_i, F^{ijk}_{lmn}, \delta_{ijk})$. In the case of discrete gauge theories, D is simply the number of elements in the gauge group.

We call the quantity $(S_1 - S_2) - (S_3 - S_4)$ the "topological entropy," $-S_{\text{top}}$, since it measures the entropy associated with the topological entanglement in Ψ. The above result implies that S_{top} is universal: it only depends on the type of topological order encoded in Ψ.

Physical picture.—The idea behind (1) is that topologically ordered states contain nonlocal entanglement. Consider, for example, the spin-1/2 model in [11] with spins located on the links $\boldsymbol{i}$ of the honeycomb lattice and with a Hamiltonian realizing a Z_2 lattice gauge theory. The ground state Ψ is known exactly. The easiest way to describe Ψ is in terms of strings [12]. One can think of each spin state as a string state, where a $\sigma^x_{\boldsymbol{i}} = -1$ spin corresponds to a link occupied by a string and a $\sigma^x_{\boldsymbol{i}} = 1$ spin corresponds to an empty link. In this language, Ψ is simple: $\Psi(X) = 1$ for string states X where the strings form closed loops, and Ψ vanishes otherwise.

All local correlations $\langle \sigma^x_{\boldsymbol{i}} \sigma^x_{\boldsymbol{j}} \rangle$ vanish for this state. However, Ψ contains *nonlocal* correlations. To see this, imagine drawing a curve C in the plane (see Fig. 2). There is a nonlocal correlation between the spins on the links crossing this curve: $\langle W(C) \rangle = \langle \prod_{i \in C} \sigma^x_{\boldsymbol{i}} \rangle = 1$. This correlation originates from the fact that the number of strings crossing the curve is always even. Similar correlations

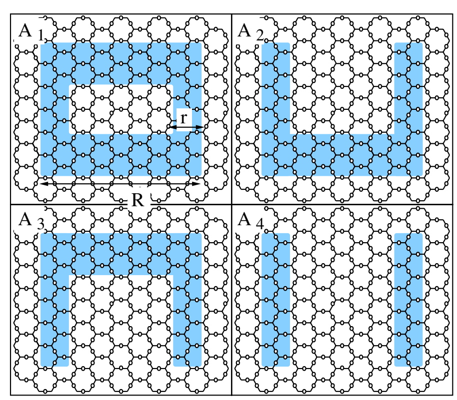


FIG. 1 (color online). One can detect topological order in a state Ψ by computing the entanglement entropies S_1, S_2, S_3, S_4 of the above four regions, A_1, A_2, A_3, A_4. Here the four regions are drawn in the case of the honeycomb lattice. Note that these regions have been carefully designed so that A_1 differs from A_2 in the same way that A_3 differs from A_4.

FIG. 2 (color online). The state Ψ contains nonlocal correlations originating from the fact that strings always cross a curve C an even number of times. These correlations can be measured by a string operator $W(C)$ (blue curve). For more general states, a fattened string operator $W_{\text{fat}}(C)$ (blue region) is necessary.

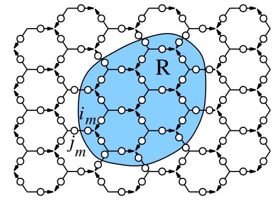


FIG. 3 (color online). A simply connected region R in the honeycomb lattice. We split the sites on the boundary links into two sites labeled $\boldsymbol{i_m}$ and $\boldsymbol{j_m}$, where $m = 1, \ldots, n$.

exist for more general states that contain virtual string-breaking fluctuations. In the general case, the nonlocal correlations can be captured by "fattened string operators" $W_{\text{fat}}(C)$ that act on spins within some distance l of C where l is the length scale for string breaking.

To determine whether a state is topologically ordered, one has to determine whether the state contains such nonlocal correlations or entanglement. While it is difficult to find the explicit form of the fattened string operators W_{fat} [13], one can establish their existence or nonexistence using quantum information theory. The idea is that if the string operators exist, then the entropy of an annular region (such as A_1 in Fig. 1) will be lower than one would expect based on local correlations.

The combination $(S_1 - S_2) - (S_3 - S_4)$ measures exactly this anomalous entropy. To see this, notice that $(S_1 - S_2)$ is the amount of additional entropy associated with closing the region A_2 at the top. Similarly, $(S_3 - S_4)$ is the amount of additional entropy associated with closing the region A_4 at the top. If Ψ has only local correlations with correlation length ξ then these two quantities are the same up to corrections of order $O(e^{-R/\xi})$, since A_2, A_4 only differ by the region at the bottom. For such states, $\lim_{R\to\infty}(S_1 - S_2) - (S_3 - S_4) = 0$. Thus, a *nonzero* value for S_{top} signals the presence of nonlocal correlations and topological order.

The universality of S_{top} can also be understood from this picture. Small deformations of Ψ will typically modify the form of the string operators W_{fat} and change their width l. However, as long as l remains finite, $(S_1 - S_2) - (S_3 - S_4)$ will converge to the same value when the width r of the annular region is larger than l.

A simple example.—Let us compute the topological entropy of the ground state Ψ of the Z_2 model and confirm (1) in this case. We will first compute the entanglement entropy S_R for an arbitrary region R. To make the boundary more symmetric, we split the sites on the boundary links into two sites (see Fig. 3). The wave function Ψ generalizes to the new lattice in the natural way.

We will decompose Ψ into $\Psi = \sum_l \Psi_l^{\text{in}} \Psi_l^{\text{out}}$ where Ψ_l^{in} are wave functions of spins inside R, Ψ_l^{out} are wave functions of spins outside R, and l is a dummy index. A simple decomposition can be obtained using the string picture. For any $q_1, \ldots, q_n$, with $q_m = 0, 1$, and $\sum_m q_m$ even, we can define a wave function $\Psi^{\text{in}}_{q_1,\ldots,q_n}$ on the spins inside of R: $\Psi^{\text{in}}_{q_1,\ldots,q_n}(X) = 1$ if (a) the strings in X form closed loops and (b) X satisfies the boundary condition that there is a string on $\boldsymbol{i_m}$ if $q_m = 1$, and no string if $q_m = 0$. Similarly, we can define a set of wave functions $\Psi^{\text{out}}_{r_1\ldots,r_n}$ on the spins outside of R.

If we glue Ψ^{in} and Ψ^{out} together—setting $q_m = r_m$ for all m—the result is Ψ. Formally, this means that

$$\Psi = \sum_{q_1+\cdots+q_l \text{ even}} \Psi^{\text{in}}_{q_1,\ldots,q_n} \Psi^{\text{out}}_{q_1\ldots,q_n}. \tag{2}$$

It is not hard to see that the functions $\{\Psi^{\text{in}}_{q_1,\ldots,q_n}: \sum_m q_m \text{even}\}$, and $\{\Psi^{\text{out}}_{r_1\ldots,r_n}: \sum_m r_m \text{even}\}$ are orthonormal. Therefore, the density matrix for the region R is an equal weight mixture of all the $\{\Psi^{\text{in}}_{q_1\ldots,q_n}: \sum_m q_m \text{even}\}$. There are 2^{n-1} such states. The entropy is therefore $S_R = (n-1)\log 2$ [8].

This formula applies to simply connected regions like the one in Fig. 3. The same argument can be applied to general regions R and leads to $S_R = (n-j)\log 2$, where n is the number of spins along ∂R, and j is the number of disconnected boundary curves in ∂R.

We are now ready to calculate the topological entropy associated with Ψ. According to (1) we need to calculate the entropy of the four regions shown in Fig. 1. From $S_R = (n-j)\log 2$, we find $S_1 = (n_1 - 2)\log 2$, $S_2 = (n_2 - 1) \times \log 2$, $S_3 = (n_3 - 1)\log 2$, and $S_4 = (n_4 - 2)\log 2$, where n_1, n_2, n_3, n_4 are the number of spins along the boundaries of the four regions. The topological entropy is therefore $-S_{\text{top}} = (n_1 - n_2 - n_3 + n_4 - 2)\log 2$. But the four regions are chosen such that $(n_1 - n_2) = (n_3 - n_4)$. Thus the size dependent factor cancels out and $-S_{\text{top}} = -2\log 2 = -\log(2^2)$. This agrees with (1) since the total quantum dimension of Z_2 gauge theory is $D = 2$.

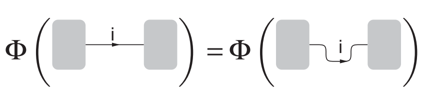
General string-net models.—To derive (1) in the general (parity invariant) case, we compute the topological entropy for the exactly soluble string-net models discussed in Ref. [10]. The ground states of these models describe all $(2+1)$-dimensional parity invariant topological orders. The models and the associated topological orders are characterized by several pieces of data: (a) an integer N—the number of string types. (b) A completely symmetric tensor δ_{ijk} where $i, j, k = 0, 1, \ldots, N$ and δ_{ijk} only takes on the values 0 or 1. This tensor represents the branching rules:

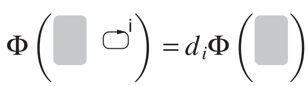
three string types i, j, k are allowed to meet at a point if and only if $\delta_{ijk} = 1$. (c) A dual string type i^* corresponding to each string type i. This dual string type corresponds to the same string, but with the opposite orientation. (d) A real tensor d_i and a complex tensor F^{ijm}_{kln} satisfying certain algebraic relations [10]. For each set of F^{ijm}_{kln}, d_i, δ_{ijk} satisfying these relations, there is a corresponding exactly soluble topologically ordered spin model.

The spins in the model are located on the links $\boldsymbol{k}$ of the honeycomb lattice. However, the spins are not usual spin-$1/2$ spins. Each spin can be in $N+1$ different states which we will label by $i = 0, 1, \ldots, N$. The Hamiltonian of the model involves a 12 spin interaction [10]. The model is known to be gapped and topologically ordered and all the relevant quantities—ground state degeneracies, quasiparticle statistics, etc. can be calculated explicitly.

The ground state wave function Φ is also known exactly. It is easiest to describe using the string-net language. One first needs to pick an orientation for each link on the honeycomb lattice. When a spin is in state i, we think of the link as being occupied by a type-i string oriented in the appropriate direction. If a spin is in state $i = 0$, then we think of the link as empty. In this way spin states correspond to string-net states (see Fig. 4).

If a spin configuration $\{i_{\boldsymbol{k}}\}$ corresponds to an invalid string-net configuration—that is, a string-net configuration that does not obey the branching rules defined by δ_{ijk}—then $\Phi(\{i_{\boldsymbol{k}}\}) = 0$. On the other hand, if $\{i_{\boldsymbol{k}}\}$ corresponds to a valid string-net configuration then the amplitude is in general nonzero. What are these amplitudes? Unfortunately, we do not have an explicit formula. However, we can write down linear relations that determine the amplitudes uniquely. These relations relate the amplitudes of string-net configurations that only differ by small local transformations. The relations are given by


(3)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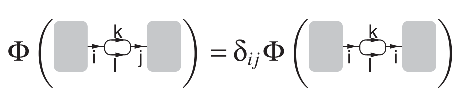
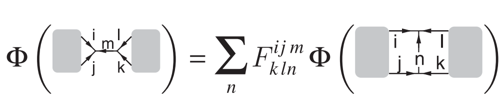

(5)


(6)

where the shaded areas represent other parts of the string nets that are not changed. Also, the type-0 string is interpreted as the no-string (or vacuum) state. The first relation (3) is drawn schematically. The more precise statement of this rule is that any two string-net configurations on the honeycomb lattice that can be continuously deformed into each other have the same amplitude. In other words, the

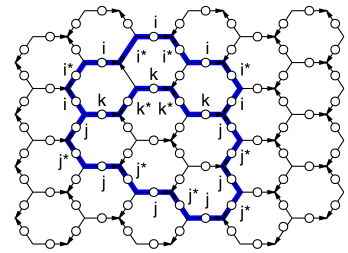


FIG. 4 (color online). A typical string-net state on the honeycomb lattice. The empty links correspond to spins in the $i = 0$ state. The orientation conventions on the links are denoted by arrows.

string-net wave function Φ only depends on the topologies of the network of strings.

By applying these relations multiple times, one can compute the amplitude for any string-net configuration (on the honeycomb lattice) in terms of the amplitude of the vacuum configuration. Thus, (3)–(6) completely specify the ground state wave function Φ.

Let us first compute the von Neumann entropy S_R of the exact ground state wave function Φ for a simply connected region R (see Fig. 3). Again we split the site on the boundary links into two sites. We decompose Φ into $\Phi = \sum_l \Phi^{\text{in}}_l \Phi^{\text{out}}_l$, where Φ^{in}_l are wave functions of spins inside R, Φ^{out}_l are wave functions of spins outside R, and l is some dummy index.

A wave function Φ^{in} on the spins inside of R can be defined as follows. Let $\{i_{\boldsymbol{k}}\}$ be some spin configuration inside of R. If $\{i_{\boldsymbol{k}}\}$ does not correspond to a valid string-net configuration—that is, one that obeys the branching rules, then we define $\Phi^{\text{in}}(\{i_{\boldsymbol{k}}\}) = 0$. If $\{i_{\boldsymbol{k}}\}$ does correspond to a valid string-net configuration, then we define $\Phi^{\text{in}}(\{i_{\boldsymbol{k}}\})$ using the graphical rules (3)–(6).

However, there is an additional subtlety. Recall that in the case of Φ, the graphical rules could be used to reduce any string-net configuration to the vacuum configuration. To fix Φ, we defined $\Phi(\text{vacuum}) = 1$.

In this case, since we are dealing with a region R *with a boundary*, string-net configurations cannot generally be reduced to the vacuum configuration. However, they can be reduced to the treelike diagrams $X_{\{q,s\}}$ shown in Fig. 5(a). Thus, to define Φ^{in}, we need to specify the amplitude for all of these basic configurations. There are multiple ways of doing this and hence multiple possibilities for Φ^{in}. Here, we will consider all the possibilities. For any labeling $q_1, \ldots, q_n, s_1, \ldots, s_{n-3}$ of the string-net in Fig. 5(a), we define a wave function $\Phi^{\text{in}}_{\{q,s\}}$ by $\Phi^{\text{in}}_{\{q,s\}}(X_{\{q',s'\}}) = \delta_{\{q\},\{q'\}}\delta_{\{s\},\{s'\}}$. Starting from these amplitudes and using the graphical rules (3)–(6) we can determine $\Phi^{\text{in}}_{\{q,s\}}(X)$ for all other string-net configurations. In the same way, we can define wave functions $\Phi^{\text{out}}_{\{r,t\}}$ on the spins outside of R through $\Phi^{\text{out}}_{\{r,t\}}(Y_{\{r',t'\}}) = \delta_{\{r\},\{r'\}}\delta_{\{t\},\{t'\}}$, where the $Y_{\{r,t\}}$ are shown in Fig. 5(b).

Now consider the product wave functions $\Phi^{\text{in}}_{\{q,s\}}\Phi^{\text{out}}_{\{r,t\}}$. These are wave functions on the all the spins in the sys-

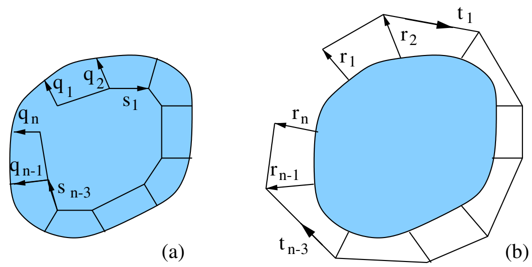


FIG. 5 (color online). The basic string-net configurations (a) $X_{\{q,s\}}$ for inside R and (b) $Y_{\{r,t\}}$ for outside R.

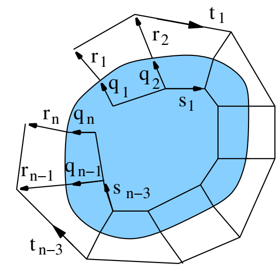


FIG. 6 (color online). The string-net configuration $Z_{\{q,s,r,t\}}$ obtained by "gluing" the configuration $X_{\{q,s\}}$ to $Y_{\{r,t\}}$.

tem—both inside and outside R. They can be generated from the amplitudes for the string-net configurations $Z_{\{q,s,r,t\}}$ in Fig. 6:

$$\Phi^{\rm in}_{\{q,s\}}\Phi^{\rm out}_{\{r,t\}}(Z_{\{q',s',r',t'\}}) = \delta_{\{q\},\{q'\}}\delta_{\{s\},\{s'\}}\delta_{\{r\},\{r'\}}\delta_{\{t\},\{t'\}}.$$

On the other hand, it is not hard to show that for the ground state wave function Φ, the amplitude for $Z_{\{q,s,r,t\}}$ is

$$\Phi(Z_{\{q,s,r,t\}}) = \delta_{\{q\},\{r\}}\delta_{\{s\},\{t\}}\prod_m(\sqrt{d_{q_m}}).$$

Comparing the two, we see that

$$\Phi = \sum_{\{q,s,r,t\},} \Phi^{\rm in}_{\{q,s\}}\Phi^{\rm out}_{\{r,t\}}\delta_{\{q\},\{r\}}\delta_{\{s\},\{t\}}\prod_m(\sqrt{d_{q_m}}). \qquad (7)$$

It turns out that the wave functions $\{\Phi^{\rm in}_{\{q,s\}}\}$ are orthonormal, as are the $\{\Phi^{\rm out}_{\{r,t\}}\}$. This means that we can use them as a basis. If we denote $\Phi^{\rm in}_{\{q,s\}}\Phi^{\rm out}_{\{r,t\}}$ by $|\{q, s, r, t\}\rangle$, then in this basis, the wave function Φ is

$$\langle\{q, s, r, t\}|\Phi\rangle = \delta_{\{q\},\{r\}}\delta_{\{s\},\{t\}}\prod_m(\sqrt{d_{q_m}}). \qquad (8)$$

The density matrix for the region R can now be obtained by tracing out the spins outside of R, or equivalently, tracing out the spin states $|\{r, t\}\rangle$:

$$\langle\{q', s'\}|\rho_R|\{q, s\}\rangle = \delta_{\{q\},\{q'\}}\delta_{\{s\},\{s'\}}\prod_m d_{q_m}. \qquad (9)$$

Since the density matrix is diagonal, we can easily obtain the entanglement entropy for S_R. Normalizing ρ_R so that $\mathrm{Tr}(\rho_R) = 1$, and taking $-\mathrm{Tr}\rho_R\log\rho_R$, we find

$$S_R = -\sum_{\{q,s\}}\frac{\prod_m d_{q_m}}{D^{n-1}}\log\left(\frac{\prod_l d_{q_l}}{D^{n-1}}\right), \qquad (10)$$

where $D = \sum_k d_k^2$. The sum can be evaluated explicitly [with the help of the relations in [10]]. The result is

$$S_R = -\log(D) - n\sum_{k=0}^N\frac{d_k^2}{D}\log\left(\frac{d_k}{D}\right). \qquad (11)$$

This result applies to simply connected regions like the one shown in Fig. 1. The same argument can be applied to general regions R. In the general case, we find

$$S_R = -j\log(D) - n\sum_{k=0}^N\frac{d_k^2}{D}\log\left(\frac{d_k}{D}\right), \qquad (12)$$

where n is the number of spins along ∂R, and j is the number of disconnected boundary curves in ∂R.

We can now calculate the topological entropy associated with Φ. Applying (12), we find $S_1 = -2\log D - n_1 s_0$, $S_2 = -\log D - n_2 s_0$, $S_3 = -\log D - n_3 s_0$, and $S_4 = -2\log D - n_4 s_0$, where n_1, n_2, n_3, n_4 are the numbers of spins along the boundaries of the four regions, and $s_0 = \sum_{k=0}^N\frac{d_k^2}{D}\log(\frac{d_k}{D})$. The topological entropy is therefore $-S_{\rm top} = -2\log D + (n_1 - n_2 - n_3 + n_4)s_0 = -2\log D$, in agreement with (1).

This research is supported by NSF Grant No. DMR-01-23156 and by ARO Grant No. W911NF-05-1-0474.

Note added.—Near the completion of this Letter, we become aware of a similar result, obtained independently in the recent paper, Ref. [14].

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