Space-Charge Effects in the Current-Filamentation or Weibel Instability

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We consider how an unmagnetized plasma responds to an incoming flux of energetic electrons. We assume a return current is present and allow for the incoming electrons to have a different transverse temperature than the return current. To analyze this configuration we present a nonrelativistic theory of the current-filamentation or Weibel instability for rigorously current-neutral and nonseparable distribution functions, $f_0(p_x, p_y, p_z) \neq f_x(p_x)f_y(p_y)f_z(p_z)$. We find that such distribution functions lead to lower growth rates because of space-charge forces that arise when the forward-going electrons pinch to a lesser degree than the colder, backward-flowing electrons. We verify the growth rate, range of unstable wave numbers, and the formation of the density filaments using particle-in-cell simulations.

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A fundamental issue in plasma physics is how a plasma responds to an externally applied electric field. Another closely related issue is how a plasma responds to an incoming flux of energetic electrons. In this case, a return current is generated, resulting in essentially no net current at each local position; however, there is net heat flux. The total system is then current-neutral with an anisotropic velocity distribution function. It is the purpose of this Letter to investigate the conditions under which the incoming flux of electrons will filament in a collisionless plasma. Apart from being of interest as a basic plasma physics problem [1–3], this current-filamentation instability is of practical importance for various phenomena in the laboratory as well as astrophysics. Understanding this instability in the context of fast ignition [4] is of particular interest because it is hoped that the magnetic fields resulting from it will focus and guide the incoming electrons generated by an intense laser.

Since the seminal work of Weibel [1], it has been well-known that a separable anisotropic bi-Gaussian velocity distribution function, $f_0(v_z, v_\perp) = [(2\pi)^{3/2} v_{\text{th}z} v_{\text{th}\perp}^2]^{-1} \times \exp[-v_z^2/(2v_{\text{th}z}^2) - v_\perp^2/(2v_{\text{th}\perp}^2)]$, is stable for electrostatic perturbations but unstable for purely electromagnetic ones. It is also well-known that a beam (bump on tail) carrying a current through a plasma will filament by a current-filamentation instability that has a physical mechanism closely related to the Weibel instability [3,5]. If a mode is purely electromagnetic then no electron density filaments can exist. This implies the presence of a neutralizing return current, which filaments at the same rate but in the opposite sense because the magnetic force is in the opposite direction.

Cold fluid theory has been employed to study the electrostatic coupling to the purely electromagnetic mode [6], and ion motion associated with this mode for cold beams has been observed in 2D particle-in-cell (PIC) simulations [7], indicating that coupling may be significant.

If the current is carried by a beam then the system is also susceptible to two-stream instabilities. The two-stream instability combines with the current-filamentation mode to yield the fastest-growing mode in a direction with a wave vector at an angle with respect to the beam drift [8,9]. This mode is a mixture of longitudinal and transverse fields. Therefore, electron density filaments and space-charge fields can be present. Bret *et al.* [9] argued that this coupling explains the electron density filaments observed in recent experiments [10].

In this Letter, we offer a different explanation and we argue that, in general, current filamentation leads to nonoscillatory density filaments. In most situations of interest the forward-flowing electrons (in the $\hat{\boldsymbol{e}}_z$ direction in our configuration) do not form a bump on tail but rather a distribution with monotonically decreasing tail [11]. Furthermore, the transverse temperatures (T_x and/or T_y) for the forward- and backward-flowing electrons differ, meaning the total distribution function is not separable. Under these circumstances the forward and returning electrons do not pinch (i.e., filament) at the same rate from the magnetic pinching forces. This results in density filaments and hence electrostatic fields that reduce the growth rate and also cause the ions to respond.

We assume that the electron velocity distribution function can be approximated by a sum of drifting (in the $\hat{\boldsymbol{e}}_z$ direction) Maxwellians, that the ions form a nondrifting cold background and rigorous current and space-charge neutrality. We will consider systems that are stable to two-stream modes in the $\hat{\boldsymbol{e}}_z$ direction. In our model the distribution functions for electrons and ions are described by

$$f_{0J}(\boldsymbol{v}) = \frac{n_{0J}}{2\pi v_{tJ}^2} e^{-v_x^2/2v_{tJ}^2} e^{-(v_z - v_{dJ})^2/2v_{tJ}^2}, \qquad (1)$$

$$f_{0i}(\boldsymbol{v}) = n_{0i}\delta(v_x)\delta(v_z), \quad (2)$$

where for each electron beam ($J = 1, 2, \ldots$), v_{tJ} is the thermal velocity and v_{dJ} is the drift velocity in the z direction. We choose x to be the direction of the unstable component of the wave number ($k_x \gg k_y, k_z$). The polarization of the unstable electromagnetic wave is $\boldsymbol{E}_{\rm EM} = E_z\hat{\boldsymbol{e}}_z$, and we allow for electrostatic fields to exist ($\boldsymbol{E}_{\rm ES} = E_x\hat{\boldsymbol{e}}_x$) so as to study the coupling to the unstable mode. The instability is fed by the $\boldsymbol{v}\times\boldsymbol{B}_{\rm EM} = B_y(\boldsymbol{v}\times\hat{\boldsymbol{e}}_y)$ force and therefore the components of the distribution functions in the v_y are irrelevant and do not appear in (1) and (2).

For the coupled electromagnetic-electrostatic waves we assume perturbations $\propto \exp[i(k_x x - \omega t)]$. The linearized, Fourier-transformed Maxwell-Vlasov equations yield the dispersion relation:

$$0 = \mathrm{Det}\left\{(\omega^2 - \omega_{pi}^2 - \sum_J \omega_{pJ}^2)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}c^2k_x^2 & 0\\ 0 & 0\end{pmatrix} + \sum_J \omega_{pJ}^2\begin{pmatrix}-(\zeta_J^2+\frac{1}{2})Z_J' & \zeta_J Z_J + \frac{\zeta_J}{2}Z_J''\\ \zeta_J Z_J + \frac{\zeta_J}{2}Z_J'' & -\frac{3}{2}Z_J' - \frac{1}{4}Z_J'''\end{pmatrix}\right\} \quad (3)$$

where the plasma dispersion function is defined as

$$Z_J = Z(\xi_J) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dt \frac{e^{-t^2}}{t - \xi_J} \quad (4)$$

for $\Im(\xi_J) > 0$ and is analytically continued into the lower half plane. The nonrelativistic plasma frequencies are defined as $\omega_{pJ}^2 = 4\pi n_{0J}e^2/m_e$, while the ion frequency is $\omega_{pi}^2 = 4\pi n_{0i}e^2/M_i$. Additionally we define $\xi_J = \omega/(\sqrt{2}k_x v_{tJ})$, $\zeta_J = v_{dJ}/(\sqrt{2}v_{tJ})$. The number of primes on the Z functions denotes the number of differentiations. If we set the two diagonal terms equal to zero, for $\xi_J \ll 1$ we obtain the dispersion relation for electromagnetic modes in the kinetic limit [2] and the ion acoustic branch, respectively. The nondiagonal terms lead to coupling between the two types of waves. We use the power series expansion of the Z function and its derivatives to derive the dispersion relation from (3) in the limit of weak anisotropy ($\xi_J, \zeta_J \ll 1 \forall\ J$). In other words, assuming $v_{dJ} \ll v_{tJ}$ as well as ω_{pJ}, $k_x v_{tJ} \gg \omega, \omega_{pi}$ and dropping the displacement current, the dispersion relation may be written as

$$0 \simeq \mathrm{Det}\begin{pmatrix} iE_1\omega + E_0 & C_1\omega\\ C_1\omega & I_2\omega^2 - \omega_{pi}^2\end{pmatrix}, \quad (5)$$

$$\simeq \omega^2\{i[I_2E_1]\omega + [I_2E_0 - C_1^2]\} - \omega_{pi}^2\{iE_1\omega + E_0\}, \quad (6)$$

where we define $E_1 = \sqrt{\frac{\pi}{2}}\sum_J \frac{\omega_{pJ}^2}{k_x v_{tJ}}$, $E_0 = \sum_J \omega_{pJ}^2 \frac{v_{dJ}^2}{v_{tJ}^2} - c^2k_x^2$, $I_2 = \sum_J \frac{\omega_{pJ}^2}{k_x^2 v_{tJ}^2}$, $C_1 = \sum_J \omega_{pJ}^2 \frac{v_{dJ}}{k_x v_{tJ}^2}$. Examination of the dispersion relation (6) reveals that depending on the value of ξ_J the growth rate can be approximately obtained by setting either the second or the first curly bracket term equal to zero. We obtain from (6):

$$\text{If } \omega \ll \frac{m_e}{M_i}k_x v_{tJ}, \qquad \Rightarrow \omega \simeq i\frac{E_0}{E_1}; \quad (7)$$

$$\text{If } \frac{m_e}{M_i}k_x v_{tJ}, \quad \omega_{pi} \ll \omega \ll \omega_{pJ}, \qquad \Rightarrow \omega \simeq i\frac{E_0}{E_1} - i\frac{C_1^2}{I_2E_1} \quad (8)$$

where (8) can be seen as a solution of (6) in the limit of infinitely massive ions.

In the absence of the nondiagonal terms, namely $C_1 = 0$, (8) reduces to (7), which recovers the kinetic limit of the purely electromagnetic mode [2]. Ignoring the coupling is not necessarily self-consistent as it implies that the system is initially charge neutral and that charge neutrality is not disturbed by the linear growth of the instability. If the electron beams have different temperatures, they tend to pinch to a different degree in the magnetic field of the unstable electromagnetic mode, and therefore a charge imbalance is generated. To illustrate this, let us assume that all of the beams have equal temperatures and that the system is current-neutral; we obtain

$$\text{For } v_{tJ} = v_t \forall\ J \qquad \Rightarrow C_1 = \frac{1}{k_x v_t^2}\sum_J \omega_{pJ}^2 v_{dJ} = 0 \quad (9)$$

and, therefore, there are no electrostatic effects. By expanding the distribution function in terms of Hermite-Gaussian modes [12] we can show that for a general electron distribution function that is separable $f(v_x, v_z) = f_x(v_x)f_z(v_z)$, the coupling vanishes, i.e., $C_1 = 0$.

Because the electromagnetic beam filamentation immediately leads to charge imbalance, the cold ion background needs to move to cancel the space charge. We can see this by letting $M_i \to 0 \Rightarrow \omega_{pi}^2 \to \infty$ in the ion acoustic part of (3) and then taking the limit of weak anisotropy, to obtain (7) once again. The assumption of a purely electromagnetic mode is not physically self-consistent unless the ions can respond in the same time scale as the electrons; for this reason, if the background consists of positrons, the coupling to electrostatic modes will be negligible. However, for $M_i \gg m_e$ the ions move slowly, retarding the growth rate of the instability.

Equation (6) is a valid dispersion relation as long as $\omega_{pJ}, k_x v_{tJ} \gg \omega, \omega_{pi}$, and $v_{dJ} \ll v_{tJ}$. We point out that only for extremely low growth rates the purely electromagnetic theory (7) applies and therefore is of little interest for any realistic problem. We can see this by substituting $k_x \sim \omega_p/c$ and $v_{tJ} \sim 0.1c$ to obtain $|\omega| \ll 10^{-4}\omega_p$. From (8) it can be shown that there is always a range of k_x such that $-i\omega \geq 0$, which corresponds to growth.

Electrostatic coupling depends linearly on the wave number. From (8) we have

$$\text{Coupling term}: \frac{C_1^2}{I_2 E_1} \propto \frac{1/k_x^2}{1/k_x^3} \propto k_x. \quad (10)$$

This leads to a suppression of the high wave numbers and to a shift of the fastest-growing modes toward longer wavelengths.

Equations (5)–(8) are useful in analyzing the physical picture behind the coupling. They predict the corrections in a regime where the instability is weak (i.e., the small ξ_J expansion). However, as the physical picture does not rely on the kinetic character of the analysis, it is reasonable to try to extrapolate our conclusions to regimes with higher anisotropy as well. We next examine the other analytically tractable limit, where the anisotropy of the beams is very large.

In the high-anisotropy limit we have $v_{tJ} \ll v_{dJ}$. We can use the large-argument asymptotic expansion of the plasma dispersion function $Z_J \sim -\xi_J^{-1} - \cdots$ in the dispersion relation (3). We recognize that to the first order (ξ_J^{-1}) in the asymptotic expansion, each of the nondiagonal (coupling) terms vanishes as long as $\sum_J \omega_{pJ}^2 v_{dJ} = 0$. Therefore, while the dispersion relation in the large-argument expansion limit contains space-charge terms, these are higher order corrections to the electromagnetic mode and become irrelevant. This is why space-charge terms have heretofore been deemed unimportant.

In the case of medium anisotropy, where the drift velocity is comparable to the thermal velocity, we cannot expand the plasma dispersion function. However, the coupling for such problems can be significant. To demonstrate this, we compare PIC simulations results with numerical solutions obtained from the exact nonrelativistic dispersion relation (3) for a problem with parameters such that $\xi_J \sim 1$ and $v_{dJ} \sim v_{tJ}$.

Let us assume two counterstreaming electron beams with equal densities. Additionally we let the system be current neutral (i.e., $v_{d1} = -v_{d2}$) and have nondrifting cold mobile protons. For the coupling to become significant, one of the two beams must be much colder than the other. We choose $-p_{d1} = p_{d2} = 0.2mc$ and thermal velocities $p_{t1} = 0.11mc$ and $p_{t2} = 0.28mc$. For these parameters we performed PIC simulations with the code OSIRIS [13]. The total electron distribution function, shown in Fig. 1(a), has a single local maximum and is therefore stable to electrostatic two-stream instabilities.

To generate the plot of the growth rate versus wave number $[-i\omega(k) = \gamma(k)]$, we performed 1D simulations with 2048 cells in the direction perpendicular to the drift and 364 particles per cell, so as to obtain sufficient resolution in k space. We calculated the growth rate for several modes by using the time interval $\omega_p t = 10$–80. At around $\omega_p t \simeq 80$ the growth starts decreasing and the saturation stage of the instability begins. For these parameters, $v_{d1} \simeq 1.8 v_{t1}$. We have numerically solved the exact dispersion relation (3) for the purely electromagnetic mode and for the coupled electromagnetic-electrostatic mode for mobile and immobile ions. The results are presented in Fig. 1(b).

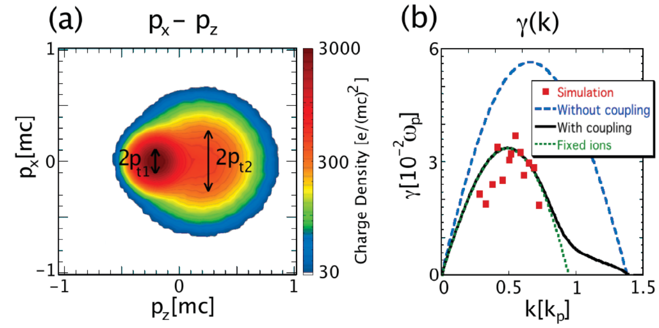

FIG. 1 (color). (a) Electron distribution function showing that the electrons are stable to two-stream instability and (b) comparison between simulation and theory illustrating the reduction of the growth rate due to space-charge effects.

The broken blue line represents the purely electromagnetic mode and the solid black line represents the correct solution, which includes electrostatic coupling and ion motion. The dotted green line shows the solution for infinitely massive ions. The red squares represent simulation results that agree with the theoretical (solid black) curve. This curve clearly illustrates that current filamentation leads to density filamentation and that the inclusion of the space-charge forces reduces both the growth rate and the wave number for the fastest-growing mode.

We have also performed 3D PIC simulations with the same beam parameters. Each electron beam is simulated by 32 particles per cell on a grid of $400 \times 400 \times 208$ cells. Periodic boundary conditions are used and the beams drift in the z direction (the one with the 208 cells). The cell sizes are $k_{p(x,y,z)}\Delta(x, y, z) = 0.25$. Unstable modes grow in each transverse (x, y) direction with identical growth rates, leading to the typical picture of current filaments pinched by the magnetic field of the unstable modes. The 3D simulation also shows that the system is two-stream stable, so unstable modes have $k_z = 0$.

Figure 2(a) shows two isosurfaces, one of the density of the hot electron beam in red ($p_{t2} = 0.28mc$) and one of the cold beam in blue ($p_{t2} = 0.11mc$) after $\omega_p t = 248.5$. For $\omega_p t < 70$, which corresponds to approximately 2.2 e foldings, no filamentary structure can be observed in the simulation (not shown here). However, clear filaments emerge later and are shown in Fig. 2(a) after 7.9 e foldings. Even though the instability is already in the saturation stage, coalescence of the filaments has not been observed yet and the number of filaments has remained constant. After this point the filaments begin to merge slowly [3,5].

It can be seen from Fig. 2(a) that in agreement with the theory, approximately 8×8 filaments fit in this $100c/\omega_p \times 100c/\omega_p$ area. In Fig. 2(b) the region outlined by the dashed box in Fig. 2(a) is blown up. It clearly shows that each filament is stable in the z direction ($k_z = 0$).

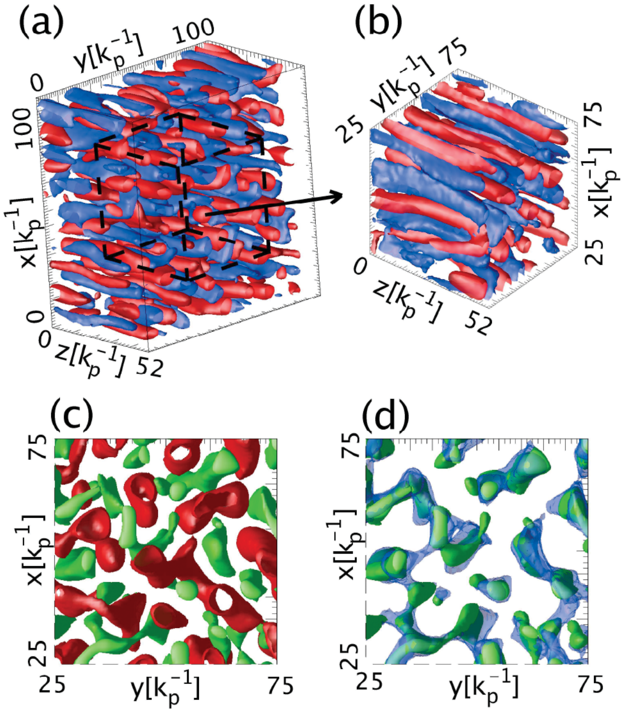

FIG. 2 (color). (a) Electron density isosurfaces for the hot beam ($p_{t2} = 0.28mc$, red) and the cold beam ($p_{t1} = 0.11mc$, blue) at $\omega_p t = 248.5$ confirm the number of filaments predicted by our theory, (b) zoomed box shows $k_z = 0$, (c) ion filaments (green) and hot beam filaments (red) are in complementary locations (zoomed box) and (d) ion filaments (green) and cold beam filaments (blue) overlap (zoomed box).

Because of the emergence of space-charge fields, the ions start moving. Ion filaments develop slowly and are shown (in green) in Figs. 2(c) and 2(d) along with the filaments of the hot beam and the cold beam, respectively. The ion density filaments follow those of the cold beam, which tends to filament more easily, in agreement with the physical picture for the coupling. The electron filaments correspond to a density 20% higher than the unperturbed value while the ion filaments to a 0.2% perturbation. This verifies that the ions do not respond fast enough to maintain charge neutrality.

The nonlinear steady state with ion filaments overlapping the electron filaments of the return current was observed in simulations for the fast ignitor concept [11] and can be explained from the space-charge coupling described above. The transfer of energy from the purely electromagnetic mode to electrostatic modes due to coupling is expected to significantly affect the development of the magnetic structures in both the linear and nonlinear stages of the instability.

This theory deals with nonrelativistic Gaussian distribution functions. Similar coupling is expected to occur for relativistic distribution functions; however, the integrals in the momentum space cannot be carried out analytically because of the coupling of the different momentum components in the relativistic limit via the γ factor. We have used relativistic waterbag distribution functions [12], which do allow analytical evaluation of the integrals [14], and terms similar to the ones discussed above do arise. However, the waterbag distribution functions are an appropriate model for the fluid and therefore high-anisotropy limit of the instability. Because the space-charge coupling becomes important at the same point in which waterbag distribution functions become an unreliable approximation, we cannot use them to make any quantitative prediction.

We have examined the effects of coupling of electrostatic modes to the electromagnetic current-filamentation instability where there is a neutralizing return current. It has been shown that the instability can be expected to grow much more slowly than purely electromagnetic instabilities and with lower wave numbers. The space-charge forces cause the ions to also filament with the ions overlapping the colder backward-flowing electrons. These effects significantly alter the development of the instability and need to be taken into account whenever the electrons are described by a nonseparable distribution function.

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