

Unified Picture of Ferromagnetism, Quasi-Long-Range Order, and Criticality in Random-Field Models

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By applying the recently developed nonperturbative functional renormalization group (FRG) approach, we study the interplay between ferromagnetism, quasi-long-range order (QLRO), and criticality in the d -dimensional random-field $O(N)$ model in the whole (N, d) diagram. Even though the “dimensional reduction” property breaks down below some critical line, the topology of the phase diagram is found similar to that of the pure $O(N)$ model, with, however, no equivalent of the Kosterlitz-Thouless transition. In addition, we obtain that QLRO, namely, a topologically ordered “Bragg glass” phase, is absent in the 3-dimensional random-field XY model. The nonperturbative results are supplemented by a perturbative FRG analysis to two loops around $d = 4$.

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How the phase behavior and ordering transitions of a system are affected by the presence of a weak random field remains in part an unsettled problem. Heuristic and rigorous arguments show that the lower critical dimension, below which no long-range order is possible, is 2 for the random-field (RF) Ising model [1,2] and 4 for models with a continuous symmetry [RFO(N)M, with $N > 1$] [1,3,4]. However, this leaves aside two questions: first, the nature of the critical behavior in random-field models, a question connected to the breakdown of the so-called “dimensional reduction” (DR) property that relates the critical exponents of the RFO(N)M to those of the pure $O(N)$ model in two dimensions less [5] and, second, the possible occurrence of a low-temperature phase with quasi-long-range order (QLRO), i.e., a phase characterized by no magnetization and a power-law decrease of the correlation functions, in models with a continuous symmetry [6,7]. Progress has been made to better circumscribe this latter point. It has indeed been shown that QLRO is absent for $N = 2$ when disorder is strong and for $N > 3$ for an arbitrarily weak random field [8], but this still keeps open the cases $N = 2, 3$ in the physical dimensions $d = 2, 3$.

Those questions are important because, on top of purely theoretical motivations, they concern the behavior of the known experimental realizations of random-field models. This is the case, for instance, of vortex lattices in disordered type-II superconductors [7,9]. In such systems, the randomly pinned lattice of vortices can be mapped onto an “elastic glass model” [7,9], whose simplest realization is the $N = 2$ RFXYM. The occurrence of a phase with QLRO, termed “Bragg glass,” has been predicted for the $3 - d$ version of the model [7]. Further theoretical support for this prediction has been given by a Monte Carlo simulation of the RFXYM [10] and by analyses of the energetics of dislocation loops [7,11].

In this Letter, we apply our recently developed nonperturbative functional renormalization group (FRG) approach of the RFO(N)M [12] to provide a unified picture of

ferromagnetism, QLRO, and criticality in the whole (N, d) diagram. We find that below a critical value $N_c = 2.83 \dots$ and for $d < 4$ the model has a transition to a QLRO phase, both this phase and the transition being governed by zero-temperature nonanalytic fixed points (FPs). The transition disappears below a lower critical dimension, which we find around 3.9 for $N = 2$. Therefore, contrary to what is usually believed, no QLRO and no Bragg glass phase exist in the $3 - d$ RFXYM. We supplement our nonperturbative, but of course approximate, results by a perturbative FRG analysis to two loops in $d = 4 + \epsilon$. The present approach allows us to discuss the DR property and its breakdown. We find, in particular, that the topology of the (N, d) phase diagram of the RFO(N)M is similar to that of the pure $O(N)$ model, with, however, no equivalent of the $(N = 2, d = 2)$ Kosterlitz-Thouless (KT) transition.

Our starting point is the standard effective Hamiltonian for the RFO(N)M in d dimensions with an N -component field $\chi(x)$ and uncorrelated random fields taken from a Gaussian distribution with zero mean and variance Δ . After introducing n replicas in order to perform the average over quenched disorder (taking at the end the limit $n \rightarrow 0$), it can be rewritten as:

$$S[\{\chi_a\}] = \int_x \left\{ \frac{1}{2T} \sum_{a=1}^n \left[\partial \chi_a \cdot \partial \chi_a + \tau \chi_a^2 + \frac{u}{12} \chi_a^4 \right] - \frac{\Delta}{2T^2} \sum_{a,b=1}^n \chi_a \cdot \chi_b - \sum_{a=1}^n \mathbf{J}_a \cdot \chi_a \right\}, \quad (1)$$

where we have introduced sources acting on each replica separately, which therefore explicitly break the permutation symmetry between the replicas.

We apply in this work the nonperturbative FRG formalism recently proposed by us for the RFO(N)M [12]. To keep the presentation short but sufficiently self-contained, we first sketch the main steps of the approach. It is based on an exact RG equation for the effective average action $\Gamma_k[\{\Phi_a\}]$. Γ_k interpolates between the bare

action, Eq. (1), and the usual effective action [i.e., the Legendre transform of the partition function associated with Eq. (1)] as the running scale k moves from the microscopic ($k = \Lambda$) to the macroscopic ($k \rightarrow 0$) scale. It is built by integrating out fluctuations with momenta larger than k , and its flow obeys an exact equation

$$\partial_t \Gamma_k[\{\Phi_a\}] = \frac{1}{2} \text{Tr} \{ \partial_t R_k(q) (\Gamma_k^{(2)}[\{\Phi_a\}, q] + \mathbb{1} R_k(q))^{-1} \}, \quad (2)$$

where ∂_t is a derivative with respect to $t = \ln(k/\Lambda)$, $\Gamma_k^{(2)}$ is the tensor formed by the second functional derivatives of Γ_k with respect to the fields $\Phi_a(q)$, $\mathbb{1}$ is the unit tensor, and the trace involves an integration over momenta as well as a sum over replica indices and N -vector components. $R_k(q)$ is the infrared cutoff introduced to suppress the low-momentum modes. In practice, solving Eq. (2) numerically requires the introduction of approximation schemes which amount to truncating the functional form of Γ_k . Guided by the physics of the problem at hand and by the mounting work on the method [13], one can then formulate a non-perturbative RG description. For the RFO(N)M, we have argued that the two main ingredients allowing study of the long-distance physics are (i) the derivative expansion, which approximates the momentum dependence of the (1PI) vertex functions [13], and (ii) the expansion in increasing number of free replica sums, which is equivalent to include increasing-order cumulants of the renormalized distribution of the quenched disorder [12].

The simplest nonperturbative FRG description of the RFO(N)M relies on the following truncation:

$$\Gamma_k[\{\Phi_a\}] = \int_x \left\{ \sum_{a=1}^n \left(\frac{1}{2} Z_{m,k} \partial \Phi_a \cdot \partial \Phi_a + U_k(\Phi_a) \right) - \frac{1}{2} \sum_{a,b=1}^n V_k(\Phi_a, \Phi_b) \right\}, \quad (3)$$

with one single wave function renormalization for all fields, $Z_{m,k}$, evaluated at the field configurations that minimize the 1-replica potential U_k (pseudo-first-order derivative expansion [13]); the 1-replica U_k and 2-replica V_k parts of the effective potential are obtained from Γ_k taken for uniform fields. Inserting Eq. (3) into Eq. (2) leads to coupled partial differential equations for the functions $U_k(\tilde{\rho})$ and $V_k(\tilde{\rho}, \tilde{\rho}', z)$, where $\tilde{\rho} = \frac{1}{2} \Phi^2$ and $z = \Phi \cdot \Phi' / \sqrt{(4\tilde{\rho}\tilde{\rho}')}$, and to a running anomalous dimension defined as $\eta_k = -\partial_t \log Z_{m,k}$. The sought FPs being at zero temperature, one also introduces a running temperature T_k and the associated exponent $\theta_k = \partial_t \log T_k$. This is most conveniently done by defining a renormalized disorder strength $\Delta_{m,k} = (2\tilde{\rho}_{m,k})^{-1} \partial_z V_k(\tilde{\rho}_{m,k}, \tilde{\rho}_{m,k}, z=1)$, where $\tilde{\rho}_{m,k}$ corresponds to the minimum of $U_k(\tilde{\rho})$. Defining then T_k as $Z_{m,k} k^2 \Delta / (\Lambda^2 \Delta_{m,k})$, one can see that it reduces to the bare temperature T at the microscopic scale Λ .

The flow equations can be expressed in a scaled form by introducing renormalized dimensionless quantities, $u_k(\rho) = T_k k^{-d} U_k(\tilde{\rho})$, $v_k(\rho, \rho', z) = T_k^2 k^{-d} V_k(\tilde{\rho}, \tilde{\rho}', z)$,

$\rho = Z_{m,k} T_k k^{-(d-2)} \tilde{\rho}$: see Eqs. (5, 6) in Ref. [12]. An expression for η_k is derived by considering the flow of the transverse component of $d/dq^2 \Gamma_k^{(2)}(q^2)$ evaluated for uniform fields and zero momentum:

$$\eta_k = 4C_d/d \{ u_{\rho\rho} [v_z - 4\rho_m^2 v_{\rho\rho'}] m_{3,1}^d (2\rho_m u_{\rho\rho}, 0) + 2u_{\rho\rho}^2 v_z [m_{2,3}^d (2\rho_m u_{\rho\rho}, 0) + m_{3,2}^d (2\rho_m u_{\rho\rho}, 0)] \}, \quad (4)$$

where $C_d^{-1} = 2^d \pi^{d/2} \Gamma(d/2)$, $u_{\rho\rho}$, v_z , and $v_{\rho\rho'}$ stand, respectively, for $\partial_\rho^2 u$, $\partial_z v$, and $\partial_\rho \partial_{\rho'} v$ evaluated for fields that minimize u_k (the subscript k has been dropped for simplicity), and $m_{p,p'}^d(w, w')$ are dimensionless threshold functions described in Ref. [13]. In Eq. (4), we have set for clarity $T_k = T = 0$, but the same FPs are reached when $T > 0$, provided (as indeed found) that $\theta = \theta_{k \rightarrow 0} > 0$. As discussed previously [12], breakdown of DR is associated with the appearance of a strong enough nonanalyticity in the renormalized 2-replica potential as the two replica fields become equal. This behavior is related to the presence of metastable states in the renormalized random potential generated along the RG flow.

We have numerically integrated the flow equations for u_k , v_k , and η_k for a variety of initial conditions and determined in this way the FPs and the associated stability behavior (and critical exponents η , $\bar{\eta} = 2 - \theta + \eta$, and for once unstable FPs, ν). To cover the whole (N, d) diagram for continuous values of N and d with a tractable computational effort, we have used an additional approximation that consists in expanding u_k and v_k around the field configuration $\rho_{m,k}$: $u_k = u_2(\rho - \rho_{m,k})^2$, $v_k = v_{00}(s) + v_{10}(s)(\rho + \rho' - 2\rho_{m,k}) + v_{20}(s)(\rho + \rho' - 2\rho_{m,k})^2 + v_{02}(s)(\rho - \rho')^2$, where $s = (\rho + \rho' - 2\sqrt{\rho\rho'}z)/2$. One should point out that the present flow equations reproduce all 1-loop perturbative results in the appropriate region of the (N, d) plane, including the FRG equation at first order in $\epsilon = d - 4$ [12].

The central results of our study are summarized on the phase diagram displayed in Fig. 1(a). In region III, there are no phase transitions; the RFO(N)M always stays disordered. In region I, there is a transition to a ferromagnetic phase at a critical point governed by a zero-temperature nonanalytic FP. The nonanalyticity is strong enough (a

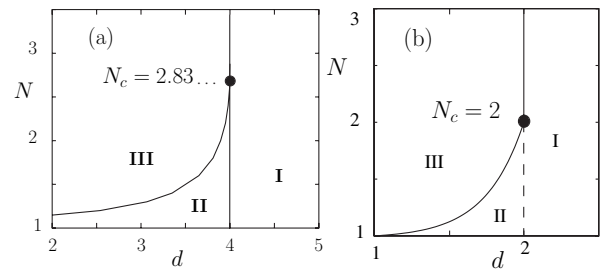


FIG. 1. Predicted phase diagram for (a) the RFO(N)M (non-perturbative FRG) and (b) pure $O(N)$ model (sketch). Regions I and II correspond to transitions to a ferromagnetic and a QLRO phase, respectively.

“cusp”) for breaking the DR predictions for the critical exponents; however, when approaching a critical line $N_{c,DR}(d)$ (not shown here, but starting from $N = 18$ for $d = 4 + \epsilon$ and going to $d \simeq 5$ when $N = 1$ [12]), the critical exponents continuously tend to their DR value. Above the line, the stable (more precisely, once unstable) FP is nonanalytic, but the singularity now only affects the corrections to scaling. (FPs with a cusp can still be found, but they are unstable in several directions and correspond therefore to multicritical points, unreachable from generic initial conditions.)

Finally, in region II, we find two zero-temperature non-analytic FPs: One is attractive and describes a QLRO phase, and the other one is once unstable and governs the transition between the disordered and the QLRO phases. The exponents η characterizing the power-law decay of the connected correlation function for the QLRO phase and the transition are shown in Fig. 2 for a range of values of N . One can see that, for N less than a critical value $N_c = 2.83\dots$ and for $d < 4$, which corresponds to region II, the two FPs coalesce for a value $d_{lc}(N)$ which then determines the lower critical dimension below which no phase transition is observed. The most striking outcome is that, contrary to what is usually believed, no QLRO, i.e., no Bragg glass phase, exists in $d = 3$ for the RFXYM: indeed, $d_{lc}(N = 2) \simeq 3.9$. Our study of the RG flow for $d = 3$ and $N = 2$ shows signatures of a pseudo-FP [14], presumably lying on the imaginary axis not too far from the physical plane of coupling constants (which could explain the behavior found in Monte Carlo simulation [10]; see Fig. 3), but the flow goes at large distance to the trivial disordered FP. We shall come back to this point later. Note that an estimate of the uncertainty of the present nonperturbative but approximate RG treatment is given by considering the point in Fig. 1(a) for $d = 2$: In this (probably most unfavorable) case, the theory predicts $N_{lc} \simeq 1.15$ instead of the exact result $N_{lc} = 1$. As shown for the pure $O(N)$ model

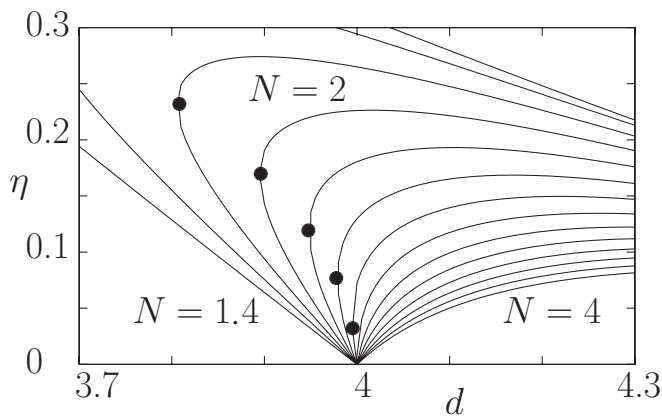


FIG. 2. Prediction for the d dependence of the anomalous dimension η for different values of N (by steps of 0.2). Below $N_c = 2.83\dots$ and for $d < 4$ two nontrivial FPs (QLRO and transition), each characterized by an exponent η , appear; they coalesce for $d = d_{lc}(N)$ (dots).

[15], this could be improved by solving the full first order of the derivative expansion.

It is instructive to contrast the behavior of the RFO(N)M with that of the pure model. The phase diagram of this latter, derived from the solution of phenomenological RG equations [16] and from known exact results, is shown in Fig. 1(b). Regions I, II, and III are the exact counterparts of those in Fig. 1(a). Indeed, and although never mentioned, QLRO exists in the pure model even aside from the special KT transition for $N = 2$ and $d = 2$. However, it occurs in an unphysical region, $1 < N < 2$, $1 < d < 2$. It remains that the topology of the RFO(N)M phase diagram is very similar to that of the pure model. Again, there is no exact DR-like property, the lower critical dimension $d_{lc}(N)$ being shifted by an N -dependent value between the two models.

Another intriguing question raised by this similarity is the possible equivalence between the ($N_c = 2.83\dots$, $d = 4$) point of the RFO(N)M and the ($N_c = 2$, $d = 2$) point of the pure model [compare Figs. 1(a) and 1(b)]. Our non-perturbative FRG solution of the former shows a behavior very reminiscent of the KT transition, but the approximation used is unable to rigorously locate a line of FPs as would occur for a KT transition [13,17]. To resolve this matter, we have considered the perturbative FRG analysis around $d = 4$. The 1-loop β function(al) for the renormalized 2-replica potential (to which our nonperturbative treatment exactly reduces near $d = 4$) can be set to zero in $d = 4$ when $N = N_c = 2.83\dots$ for arbitrary values of the disorder strength, much like it does in the pure $d = 2$ XY model for arbitrary values of the temperature; but what about the 2-loop β function? To answer this question, we have calculated (independently from the authors of the recent Ref. [18]) the β functions of the RFO(N)M around $d = 4$ to 2 loops, starting with the nonlinear-sigma model associated with Eq. (1). The calculation follows the treatment developed for the pure model [19], with the bare action expressed in terms of renormalized dimensionless

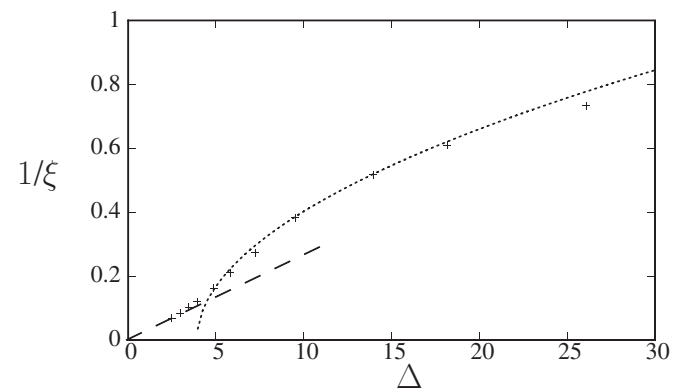


FIG. 3. Inverse correlation length vs disorder strength for the 3 - d RFXYM (arbitrary units). The crossover around $\Delta = 5$ is the remainder of the FPs found above d_{lc} . Δ parametrizes the initial conditions of the RG flow and $1/\xi$ is estimated as in Ref. [14].

quantities,

$$\begin{aligned} S[\{\boldsymbol{\pi}_a\}] = & -\frac{k^d}{2Z_T^2 t^2} \int_x \sum_{a,b=1}^n [R(Z_\pi \boldsymbol{\pi}_a \cdot \boldsymbol{\pi}_b + \sigma_a \sigma_b) \\ & + \delta[R]] + \frac{k^{d-2}}{2Z_T t} \int_x \sum_{a=1}^n \left[Z_\pi (\partial \boldsymbol{\pi}_a)^2 + (\partial \sigma_a)^2 \right. \\ & \left. - 2h \frac{Z_t}{\sqrt{Z_\pi}} \sigma_a \right], \end{aligned}$$

where the $(N-1)$ -component fields $\boldsymbol{\pi}_a$ characterize the fluctuations in the directions orthogonal to the external field h ($h = Z_\pi^{1/2} Z_T^{-1} H$), while $\sigma_a = \sqrt{1 - Z_\pi \boldsymbol{\pi}_a^2}$; $t = k^{d-2} Z_T^{-1} T$ and $R(z)$ is the renormalized 2-replica potential that we seek to relate to the bare term $R_0(z) = \Delta z$ via the counterterms $\delta[R]$, such that $R_0 = k^{4-d} (R + \delta[R])$. We then carry out the perturbation expansion around the Gaussian theory characterized by the propagator $G_{ab}^{(0)\mu\nu}(q) = t Z_T k^{2-d} (Z_\pi q^2 + Z_\pi^{1/2} Z_T h / \sigma_a)^{-1} \times (\delta_{\mu\nu} - \pi_a^\mu \pi_a^\nu) \delta_{ab}$, where $\mu, \nu = 1, \dots, N-1$. The relevant interaction vertices are obtained by taking up to six functional derivatives of the action with respect to the $\boldsymbol{\pi}_a$ fields. They involve derivatives of R , i.e., functions of z . To handle the resulting functional diagrams, we have followed the method developed for the FRG of disordered elastic systems [20]. Calculations are quite intricate and will be reproduced elsewhere.

The final expression for the $T=0$ β function for $R(z)$ can be put in the form $\partial_r R(z) = R'(1) \beta[r] = R'(1) \times (\epsilon r(z) + R'(1) \beta_1[r] + R'(1)^2 \beta_2[r])$, where $\epsilon = 4-d$, $R'(1)$ is the renormalized disorder strength, and $r(z) = R(z)/R'(1)$; the 1-loop $\beta_1[r]$ is given in Refs. [12,21] and the 2-loop contribution reads

$$\begin{aligned} 2\beta_2[r] = & y(yr''' - 3zr'' - r')^2(-yr'' + zr' - 1) \\ & - (N-2)[y^2 r'^3 - y(3zr' + y - 3)r'^2 \\ & - 2y(r' - z)r'r'' + yr'^2 + 4r] \\ & + a^2[(N+2)yr'' - (3N-2)zr' + 8K(N-2)r], \end{aligned}$$

where $y = 1 - z^2$ and $a = \lim_{(z \rightarrow 1^-)} (1 - z^2)^{1/2} r''(z)$. One can check that, with the change of variable $z = \cos\phi$ and with $K = 2\gamma_a$, the above equation is identical to the result recently obtained by Le Doussal and Wiese [18]. As expected, DR is recovered by setting the ‘‘anomalous’’ term a to zero. The coefficient K in the above expression is left so far as an unknown: It requires additional calculations (in progress) in order to fully determine the 2-loop β functions. However, its value is not needed to answer the above question, i.e., to know if for $N_c = 2.83 \dots$ and $d = 4$ the β function for $R(z)$ can vanish identically to two loops for arbitrary disorder strength $R'(1)$. This latter property is true only if both $\beta_1[r]$ and $\beta_2[r]$ vanish identically for the same function $r(z)$. One can check that this is impossible, irrespective of the value of K . We can thus conclude that the special point ($N_c = 2.83 \dots$, $d = 4$) does *not* correspond to a KT transition. Actually, as also obtained in our non-

perturbative approach [see Fig. 1(a)], the only FP found for the $(N_c, d = 4)$ model is at zero renormalized disorder strength, which allows a perturbative analysis of the two nonanalytic fixed points for $d < 4$ and $N < N_c$ [18].

One further consequence of the absence of a KT transition is that, contrary to what occurs in the pure model near $(N = 2, d = 2)$, the line of lower critical dimension approaches the $(N_c, d = 4)$ point with an infinite slope [compare Figs. 1(a) and 1(b)]. This reinforces our finding that $d = 3$ is safely below the lower critical dimension of the RFXYM. Whether or not this conclusion implies that no Bragg glass phase should exist in disordered high- T_c superconductors is, however, an open question. The mapping from these latter to the RFXYM is valid only at low disorder where an essentially elastic description can be used [22]. It may well be that the mechanism by which QLRO is destroyed in the RFXYM, a mechanism that does not explicitly invoke the presence of dislocations, is specific to that model.

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