Impurity Pinch from a Ratchet Process

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A ratchet-type average velocity is shown to appear for test particles moving in a stochastic potential and a magnetic field that is space dependent. This is a possible explanation for impurity behavior in tokamak plasmas.

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Impurity control in magnetically confined plasmas is a very important issue for the development of fusion reactors. A considerable experimental effort (see, e.g., [1–3]) has led to the conclusion that this process is far from being understood on the basis of the existing theoretical models. In particular, the experimental results show the accumulation of the impurities in the central region of the plasma, which appears to be a directed transport (a pinch) rather than a diffusive one [4]. Several models have been proposed [5–8], which explain some aspects of this behavior. We present here an alternative mechanism, which shows that an average velocity is produced in turbulent plasmas through a ratchet-type process due to the space variation of the confining magnetic field.

The so-called ratchet process [9] is a generic name for a large class of average stochastic velocities that are generated by unbiased noise. This name suggests the motion of a circular saw with asymmetric saw teeth. The ingredients of a minimal model that produces such directed transport consists of a periodic potential (or velocity) with broken reflection symmetry, and a noise (usually a Gaussian white noise). It was shown that, in spite of the asymmetry, the stochastic motion has no systematic preferential direction such that the average velocity is transitory and becomes zero at large times. But, if this kind of equilibrium is broken, a ratchet process appears. Thus, a third element has to be included in the minimal model, which can be for instance a driving force, a periodic or stochastic time variation of the amplitude of the periodic potential or of the noise, or another noise with different temperature.

We consider in slab geometry an electrostatic turbulence represented by an electrostatic potential $\phi^e(\mathbf{x}, t)$, where $\mathbf{x} \equiv (x_1, x_2)$ are the Cartesian coordinates in the plane perpendicular to the confining magnetic field. The latter is directed along z axis, $\mathbf{B} = B\mathbf{e}_z$ with $B = B_0 R/(R + x_1)$, where B_0 is the value of the magnetic field in the origin of the coordinates that is at the distance R from the symmetry axis. There is a gradient of the magnetic field ∇B along x_1 axis. The test particle motion in the guiding center approximation is determined by

$$\frac{d\mathbf{x}(t)}{dt} = v(\mathbf{x}, t) \equiv -\nabla\phi(\mathbf{x}, t) \times \mathbf{e}_z\left(1 + \frac{x_1}{R}\right), \qquad (1)$$

where $\mathbf{x}(t)$ represent the trajectory of the particle guiding center, ∇ is the gradient in the (x_1, x_2) plane and $\phi(\mathbf{x}, t) = \phi^e(\mathbf{x}, t)/B_0$. The electrostatic potential $\phi(\mathbf{x}, t)$ is considered to be a stationary and homogeneous Gaussian stochastic field, with zero average. It is completely determined by the two-point Eulerian correlation function (EC), $E(\mathbf{x}, t)$, defined by

$$E(\mathbf{x}, t) \equiv \langle \phi(\mathbf{x}', t')\phi(\mathbf{x}' + \mathbf{x}, t' + t)\rangle. \qquad (2)$$

The average $\langle \ldots \rangle$ is the statistical average over the realizations of $\phi(\mathbf{x}, t)$, or the space and time average over $\mathbf{x}'$ and t'. This function evidences three parameters that characterize the (isotropic) stochastic field: the amplitude, the correlation time τ_c, which is the decay time of the Eulerian correlation and the correlation length λ_c, which is the characteristic decay distance. These three parameters combine in a dimensionless Kubo number $K = \tau_c/\tau_{\rm fl}$ where $\tau_{\rm fl} = \lambda_c/V$ is the time of flight of the particles over λ_c and V is the amplitude of the stochastic velocity.

Particle motion in a stochastic potential was extensively studied [10,11] for constant magnetic fields [$R \to \infty$ in Eq. (1)]. It is well known since many years [12] that, for slowly varying or large amplitude turbulence corresponding to $K > 1$, the electric drift determines a process of dynamical trapping of the trajectories. It consists of trajectory winding on the contour lines of the potential. Important progress in the study of this nonlinear process was recently obtained. New statistical methods were developed [13,14] that permitted to determine the time dependent (running) diffusion coefficient and even the probability of displacements. It was shown that the trapping process completely changes the statistical properties of the trajectories determining memory effects, quasicoherent behavior and non-Gaussian distribution [14]. The diffusion coefficients decrease due to trapping and their scaling in the parameters of the stochastic field is modified [15,16].

We show here that the inhomogeneity of the confining magnetic field determines a directed transport (an average velocity), although the average of the right-hand side of Eq. (1) is zero. We note that trajectory trapping is related to the invariance of the potential along trajectories that ap-

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pears in the static case ($\tau_c \to \infty$). This property still appears in Eq. (1), which can be reduced to a Hamiltonian system.

Equation (1) is not similar to a typical model for a ratchet process. However it contains the three specific elements mentioned above and discussed in detail in [9]. They are all included in the stochastic drift velocity $v(\mathbf{x}, t)$ defined in Eq. (1). The periodic velocity does not appear explicitly but the solutions of Eq. (1) for a static potential are periodic functions of time that lie on the contour lines of $\phi(\mathbf{x})$. Thus, the Lagrangian velocity $\mathbf{v}[\mathbf{x}(t)]$ is periodic for any solution of Eq. (1). The space dependence of the magnetic field produces the symmetry braking of the Lagrangian velocity. The random dynamics is determined by the stochastic potential itself, which plays the role of the Gaussian noise found in the standard model for ratchets. The third element, which drives the system out of equilibrium, is the time variation of the potential. We show below that for static potentials the asymptotic average velocity is zero and that a ratchet velocity appears when the stochastic potential is time dependent. We also show that the ratchet process modeled by Eq. (1) has the property of current inversion: the sign of the average velocity depends on the parameters of the stochastic potential.

The average displacement generated by the stochastic Eq. (1) is determined using the decorrelation trajectory method, a semianalytical approach developed in [13,14]. The main idea in our method is to study the stochastic Eq. (1) in subensembles (S) of realizations of the stochastic field, which are determined by given values of the potential and of the velocity in the starting point of the trajectories:

$$\phi(\mathbf{0}, 0) = \phi^0, \qquad \mathbf{v}(\mathbf{0}, 0) = \mathbf{v}^0. \tag{3}$$

The stochastic potential and velocity, reduced in the subensemble (S) defined by condition (3) are Gaussian fields but nonstationary and nonhomogeneous, and they have space and time dependent averages. The average velocity determines an average trajectory in (S), $\mathbf{X}(t; S) \equiv \langle \mathbf{x}(t) \rangle_S$, where $\langle \ldots \rangle_S$ is the average over the realizations of the potential that belong to the subensemble (3). The average displacement for the whole set of realizations is obtained by summing up the contributions of each subensemble as

$$\langle \mathbf{x}(t) \rangle = \iint d\phi^0 d\mathbf{v}^0 P_1(\phi^0, \mathbf{v}^0; \mathbf{0}, 0) \mathbf{X}(t; S), \tag{4}$$

where $P_1(\phi^0, \mathbf{v}^0; \mathbf{0}, 0)$ is the probability density for the values of the potential and velocity in the point $(\mathbf{0}, 0)$. A similar equation can be written for the correlation of the Lagrangian drift velocity. The trajectories contained in a subensemble (S) have supplementary initial conditions besides the necessary and sufficient condition $\mathbf{x}(0) = \mathbf{0}$. They evolve on contour lines of the potential with the same value ϕ^0 which means that the corresponding paths are similar in the sense that they are curves with comparable sizes. The source of trajectory fluctuations, the fluctuation of the velocity field, is zero in $\mathbf{x} = \mathbf{0}$, $t = 0$. These aspects reduce the fluctuations of the trajectories in (S) and justify the approximation introduced in this method that consists in neglecting the fluctuations of the trajectories around the average trajectory in (S). The fluctuations were considered in [14] where it is shown that they have a very weak influence on the diffusion coefficient. Following the steps presented in detail in [15], one obtains a closed system of equations for the time dependent diffusion coefficient and the average displacement in the presence of a space-dependent magnetic field. The time is normalized with τ_{fl} and the distances with λ_c in the following equations and $\bar{R} = R/\lambda_c$.

$$D_{11}(t) = \frac{D_B}{(2\pi)^{3/2}} \int_0^\infty d\phi^0 \int_0^\infty du u^3 \exp\left(-\frac{u^2(1+p^2)}{2}\right) \times \int_0^{2\pi} d\beta \cos(\beta) X_1^0(u\theta(t), p, \beta), \tag{5}$$

$$\langle x_i(t) \rangle = \frac{2}{(2\pi)^{3/2}} \int_0^\infty d\phi^0 \int_0^\infty du u^2 \exp\left(-\frac{u^2(1+p^2)}{2}\right) \times \int_0^{2\pi} d\beta X_i^0(u\theta(t), p, \beta), \tag{6}$$

where X_1^0 is the component along x_1 of the average trajectory in (S) for a static potential that has the same space correlation as $\phi(\mathbf{x}, t)$. This average trajectory is obtained as the solution of the equation

$$\frac{dX_i^0}{dt} = -\varepsilon_{ij} \frac{\partial \Phi^S(\mathbf{X}^0)}{\partial X_j^0} \left(1 + \frac{X_1^0}{\bar{R}}\right), \tag{7}$$

where

$$\Phi^S(\mathbf{x}) = u\left(p + \cos(\beta)\frac{\partial}{\partial x_2} - \sin(\beta)\frac{\partial}{\partial x_1}\right)\mathcal{E}(\mathbf{x}). \tag{8}$$

is the average Eulerian potential in the subensemble (S) for the static case. It is determined by the parameters that define the subensemble (3) (represented by $u = |\mathbf{v}^0|$, $p = \phi^0/u$, and β, the angle of $\mathbf{v}^0$ with the x_1 axis) and by the EC of the stochastic potential (2). The latter is an input function that can be obtained from experiments or numerical simulations. We have considered here an analytical expression for the EC of the type $E(\mathbf{x}, t) = \Phi^2 \mathcal{E}(\mathbf{x}) h(t)$. The space factor $\mathcal{E}(\mathbf{x})$, which reflects the nonlinearity of the stochastic Eq. (1), is modeled by

$$\mathcal{E}(\mathbf{x}) = \frac{1}{\alpha(1 + |\mathbf{x}|^2/2)^\alpha}, \tag{9}$$

where isotropic potentials were chosen for simplicity. The parameter α determines the long distance decay of the EC and thus, when α decreases the spectrum of the potential is richer in long wavelength components. The normalization $\mathcal{E}(\mathbf{0}) = \alpha^{-1}$ was introduced in order to have the amplitude of the velocity independent on α. The time factor $h(t)$ describes the time variation of the stochastic potential. It appears in Eqs. (5) and (6) in the function $\theta(t)$ defined by $\theta(t) = \int_0^t h(\tau)d\tau$. This function is linear at small time $\theta(t) \cong t$ and saturates at the correlation time of the stochastic potential, which in these units is K, $\theta(t) \rightarrow_{t\rightarrow\infty} K$, for any integrable function h. The shape of this function has a weak influence on the average displacement and on the diffusion coefficient and we have chosen $h(t) = \exp(-t/K)$ in the calculations.

The average asymptotic velocity (the ratchet effect) is obtained as

$$\mathbf{V}^R = \lim_{t\rightarrow\infty} \frac{\langle \mathbf{x}(t)\rangle}{\theta(t)} = V\frac{\langle \mathbf{x}^{\rm st}(K)\rangle}{K} \qquad (10)$$

where $\langle \mathbf{x}^{\rm st}(t)\rangle$ is the average displacement (6) for a static potential [i.e., with $\theta(t) = t$ in Eq. (6)].

The average velocity (10) can be obtained analytically for potentials with fast time variation corresponding to $K \ll 1$. Since the potential changes before the trajectories travel over a correlation length, only the small time solution of Eq. (7) has to be determined. Integrating the small time approximation of Eq. (7), which is $dX_i^0/dt = (1 + X_1^0/\bar{R})v_i^0$ and using Eq. (6) one obtains an average displacement along ∇B

$$\langle x_1(t)\rangle = \bar{R}\left[\exp\left(\frac{\theta^2(t)}{2\bar{R}^2}\right) - 1\right]. \qquad (11)$$

The average asymptotic velocity for $K \ll 1$ is

$$V_1^R = V\frac{\bar{R}}{K}\left[\exp\left(\frac{K^2}{2\bar{R}^2}\right) - 1\right]. \qquad (12)$$

Thus, a ratchet effect appears in stochastic fields with small Kubo numbers provided that the magnetic field has a space variation (finite $\bar{R}$). The average velocity is always positive (directed to the region with smaller magnetic field). For large $\bar{R}$, $V_1^R = VK/2\bar{R}$.

When the Kubo number is not small the solution of Eq. (7) and the integrals in Eq. (6) have to be calculated numerically. The results obtained for the average velocity as a function of time are presented in Fig. 1. For a static potential (dashed line) the average velocity is transitory. It is positive at small time, then it becomes negative and decays to zero showing that the ratchet process does not appear in this case. The time variation of the stochastic potential determines a finite asymptotic value of the average velocity (solid line).

This ratchet velocity depends linearly on $\bar{R}^{-1}$ for large values of $\bar{R} \gtrsim 10$ (small values of ∇B) and a tendency of saturation appears at smaller $\bar{R}$. For large $\bar{R}$, which corresponds to tokamak plasma configuration, it can be written as

$$V_1^R = V\frac{1}{\bar{R}}f(K), \qquad (13)$$

where $f(K)$ is a dimensionless function, which is represented in Fig. 2 for the Eulerian correlation (9). This

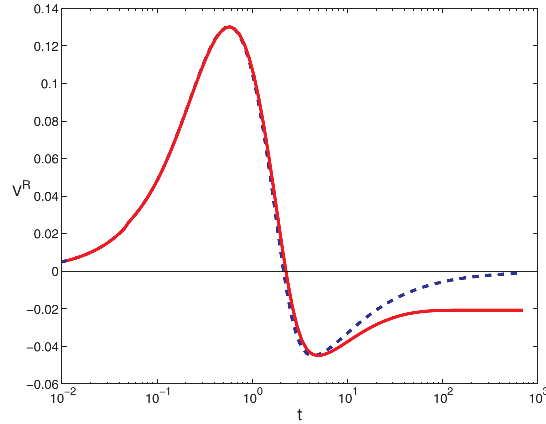

FIG. 1 (color online). The time dependent average velocity normalized with $V/\bar{R}$ for a static potential (dashed line) and for a time dependent one with $K = 20$ (solid line). $\bar{R} = 100$, and the EC is (9) with $\alpha = 1$.

function is positive for small Kubo numbers and reproduces the analytical solution (12) at $K \ll 1$. For K larger than a value $K_{\rm inv}$ that is of the order 1, the ratchet velocity becomes negative (directed to the large magnetic field region). The asymptotic K dependence is a power law $f(K) \sim -1/K^\nu$. The ratchet velocity has the same general behavior on K for any stochastic field, i.e., for any EC and, in particular, for different values of α in Eq. (9). It does not depend on the shape of the EC in the quasilinear regime ($K \ll 1$). In the nonlinear regime ($K > 1$) when trajectory trapping is effective, V^R depends on the shape of the EC and not only on K. As seen in Fig. 2, the ratchet velocity increases and $K_{\rm inv}$ decreases when α decreases. Also, the scaling parameter ν slightly depends on the shape of the EC: ν grows from 0.85 to 0.95 when α increases from 0.5 to 3. The dependence on the shape of the EC is a nonlinear effect that was found in the diffusion coefficients as well [15].

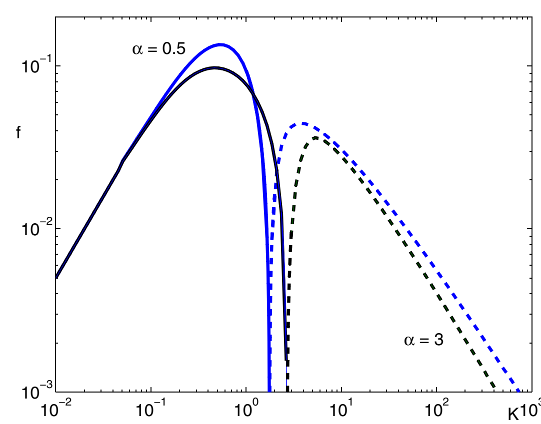

FIG. 2 (color online). The absolute value of the function $f(K)$ appearing in the ratchet velocity (13) for the EC (9) with two values of α. The negative values of f are represented by dashed lines.

The physical explanation for this behavior of the average velocity is the following. For fast variation of the stochastic field ($K \ll 1$), the displacements during the correlation time are much smaller than λ_c and they are along velocities. The latter decrease in the direction of ∇B producing displacements that are smaller than in the opposite direction. An average displacement appears in the direction $-\nabla B$ (positive x_1). At large K a part of particles are trapped and perform during τ_c almost periodic motion on contour lines of the potential. The probability of finding the particle somewhere on the contour line is proportional with the inverse of the velocity. Thus, it is larger in the high field side of each contour line and leads to an average displacement in the direction of ∇B (negative x_1). The average displacement on a potential contour line is proportional with ∇B and with the size of the contour. The probability of finding large size contour lines increases when the potentials have EC's with slower space decay (smaller α). Consequently, the ratchet velocity is larger for such potentials, in agreement with the results presented in Fig. 2.

The effect of the magnetic field inhomogeneity on the diffusion coefficients is found from Eq. (5) to be negligible at large $\bar{R}$. When $\bar{R} < 10$, the diffusion process becomes nonisotropic for $K > 1$: the component D_{11} (in the direction of ∇B) decreases, D_{22} increases and a nondiagonal diffusion coefficient appears. The scalings in K are different for different components of the diffusion tensor and continuously depend on $\bar{R}$.

The ratchet velocity obtained for typical tokamak plasma conditions is of the order of 1 m/s, which shows that it can have an important effect on the evolution of impurity density. This velocity is in the horizontal plane, along the gradient of the confining magnetic field. A similar (negative) average velocity is found in the numerical simulation of the impurity evolution in drift-Alfven turbulence [17]. Since the amplitude of the turbulence is usually larger in the small field size of the torus, the positive and the negative ratchet velocities have different effects on impurities. The positive velocity directs the impurities from the high field boundary inside the plasma but as it combines with the rotation velocity determined by the magnetic configuration, the impurities reach the exterior of the torus and they are expected to be eliminated due to the increased ratchet velocity. On the contrary, the negative ratchet velocity determines impurity penetration from the weak field side boundary and their accumulation inside plasma. Since the values of the Kubo number for tokamak turbulence are in the interval $[0.1, 10]$ both effects can appear depending on the characteristics of the turbulence. This provides in principle the possibility of controlling the impurity behavior. The ratchet velocity combines with the other sources of direct transport (curvature drift or thermodiffusion [5–8]). However, due to its specific direction (along ∇B) that determines a strong dependence of the effects on the poloidal position of the impurity source, V^R could be experimentally identified in the total average velocity.

In conclusion we have shown that the $E \times B$ stochastic drift determines a directed transport in space-dependent magnetic fields. An average velocity appears even if the turbulence is homogeneous. It is parallel or antiparallel to the gradient of the magnetic field, depending on the characteristics of the turbulence. This statistical process is not related to the curvature drift but it is a ratchet-type effect. The ratchet velocity is shown here to influence the behavior of impurities in tokamak plasmas, but, since the $E \times B$ drift is a basic nonlinearity in plasma turbulence, we expect important effects on instabilities and turbulence evolution. We have developed a nonstandard model for a ratchet process based on Eq. (1), which can have applications in other fields too. According to the classification presented in [9], it is a two-dimensional Hamiltonian stochastic ratchet. The velocity and the diffusion coefficient were obtained using a semianalytical approach, the decorrelation trajectory method.

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