

## Dynamical Control: Comparison of Map and Continuous-Flow Approaches

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Continuous and pulsed forms of control of a multistable system are compared directly, both theoretically and numerically, taking as an example the switching of a periodically driven class-B laser between its stable and unstable pulsing regimes. It is shown that continuous control is the more energy efficient. This result is illuminated by making use of the close correspondence that exists between the problems of energy-optimal control and the stability of a steady state.

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Ensuring the stability and effective control of a lasing mode represents an important problem in the applied theory of lasers [1]. It can be mapped onto analyses of spiking behavior in population dynamics [2] and neurons [3] and, in a more general context, appears as a fundamental problem in the theory of nonlinear dynamical systems [4,5]. It is potentially of relevance wherever switching takes place between distinct regimes of behavior, e.g., in cardiac and cortical systems. While the topic is thus of broad interdisciplinary interest, laser systems can provide especially reliable and convincing tests of the new theoretical concepts. In general, the problem can be analyzed within either one of two distinct theoretical and experimental frameworks: using either continuous or discrete time, with corresponding descriptions of the system dynamics in terms of either continuous flows or maps. Both methods have been extensively tested in application to laser systems. For example, a special protocol has been developed for the feedback control of Nd lasers [6]. To reduce uncertainty in switching, methods based on stochastic resonance have been proposed [7,8], with the addition of a weak periodic modulation. The targeting of stable and saddle orbits has been discussed [9] and achieved experimentally by the use of a single impulse in a loss-modulated CO<sub>2</sub> laser [10,11] paying special attention to minimization of the duration of the transient processes. Optimization of switch-on properties in semiconductor lasers has been considered by exploring phase space [12] and via the minimal-time control problem [13]. That no direct comparison between these two general approaches for control has yet been made is perhaps surprising, given its broad interdisciplinary implications.

In this Letter we consider, both theoretically and numerically, a direct comparison between continuous and discrete time approaches to control the lasing mode in class-B lasers. In particular, we show that the continuous method is the more energy efficient: the activation “energies” and the energies of the control functions differ by an order of magnitude. We use the duality of the control and stability problems discussed previously [14,15] to provide insight into the origin of this difference.

Coexistence of nonstationary states can be realized in lasers by, e.g., periodic modulation of intracavity loss [10] or pumping rate [16]. Here we study the latter, which is more suitable for class-B solid-state lasers. We start from the single-mode rate equations [16,17]:

$$\dot{u} = vu(y - 1), \quad \dot{y} = q + k \cos(\omega t) - y - yu + f(t), \quad (1)$$

where  $u$  and  $y$  are proportional to the density of radiation and carrier inversion, respectively,  $v$  is the ratio of the photon damping rate in the cavity to the rate of carrier inversion relaxation, and the cavity loss is normalized to unity. The pumping rate has a constant term  $q$  plus an external periodic modulation of amplitude  $k$ , frequency  $\omega$ ;  $f(t)$  is an additive unconstrained control function.

For class-B lasers  $v$  is large,  $v \sim 10^3$ – $10^4$ , and spiking regimes occur under deep modulation of the pumping rate. Solutions can be obtained from the corresponding two-dimensional Poincaré map [18]:

$$\begin{aligned} c_{i+1} &= q + G(c_i, \psi_i)e^{-T} + K \cos(\omega T + \psi_i) + f_i, \\ \varphi_{i+1} &= \varphi_i + \omega T, \quad \text{mod } 2\pi, \end{aligned} \quad (2)$$

where  $G(c_i, \psi_i) = c_i - g - q - K \cos \psi_i$ ,  $K = k(1 + \omega^2)^{-1/2}$ , and  $\psi_i = \varphi_i - \arctan(\omega)$ . The control function  $f_i$  is now defined in discrete time. The functions  $g = g(c_i)$  and  $T = T(c_i, \varphi_i)$  are given by nonlinear equations [18]. The variables  $c_i$ ,  $\varphi_i$  correspond to the population inversion  $y(t_i)$  and phase of modulation  $\varphi_i = \omega t_i, \text{ mod } 2\pi$ , at the instants  $t_i$  of impulse onset when  $\dot{u}(t_i) > 0$ ,  $\dot{y}(t_i) > 0$ ;  $g(c_i)$  denotes the energy of the impulse; and  $T(c_i, \varphi_i)$  gives the time interval between sequential impulses. The map has been derived by asymptotic integration of Eq. (1) to an accuracy of  $O(v^{-1})$ . It is therefore valid for  $q, k, \omega \ll v$  and  $c_i > 1 + O(v^{-1})$ .

For each iteration of the map, one can find characteristics directly comparable with experiment:  $\varphi_i$  is the phase of the modulation signal at the instant of the spike and  $T(c_i, \varphi_i)$  is the interval between impulses, the energy of the spike, and its maximal intensity being given by  $u_{\max} = 1 + v[c_i - 1 - \ln(c_i)]$  [18]. Period-1 fixed points of the map determine spiking solutions at periods  $T_n = nT_M$ ,

$n = 1, 2, \dots$ , multiple to the driving period  $T_M = 2\pi/\omega$ . By using the map (2) we determine analytically regions of multistability and we can approximate the location of saddles and stable cycles.

We now set  $v = 10000$ ,  $q = 1.9$ ,  $k = 0.75$ , and  $\omega = 71.6$  and consider controlled migration from stable cycles to saddle cycles of the same amplitude and period: specifically, from the stable cycle  $C_3$  to the saddle cycle  $S_3$ , i.e., where both cycles are of period 3 relative to the external modulation. We consider two different forms of control force  $f(t)$ : one is continuous  $f_c(t)$  in time, and the other is a sequence of discrete impulses  $f_d(t)$  applied at the instants when the system crosses the Poincaré section  $u(t_i) = 1$ ,  $\dot{u}(t_i) > 0$ , i.e., coinciding with the laser spikes;  $f_d(t) = Af_i$ , if  $t \in [t_i, t_i + \tau_c]$ , and  $f_d(t) = 0$  otherwise. To determine the force  $f_d(t)$  we first solve the control problem for the map (2) and obtain  $f_i$ . The force  $f_i$  can be reproduced in the original system (1) with a series of short impulses, keeping the position and relative amplitudes of impulses. So to transform we determine an amplitude coefficient  $A$ , such that  $f(t) = Af_i$ , as the minimum factor by which we have to multiply  $f_i$  to induce the migration between cycles in the flow system (1). For the method to be valid, the duration  $\tau_c$  of the control impulses should not exceed the duration of the laser spikes  $\sim v^{-1}$ .

We wish to solve the following energy-optimal control problem: How can the system (1) with unconstrained control function  $f_c(t)$  or  $f_i$  be steered between coexisting states such that the ‘‘cost’’ functional  $J_c$  or  $J_d$ ,

$$J_c = \inf_{f \in F} \frac{1}{2} \int_{t_0}^{t_1} f^2(t) dt, \quad J_d = \inf_{f \in F} \frac{1}{2} \sum_{i=1}^N f_i^2, \quad (3)$$

is minimized? Here  $t_1$  (or  $N$ ) is unspecified and  $F$  is the set of control functions.

The solution of this problem is in general a very complicated task. But, if a solution exists, it can [14,15,19] be identified with the solution of the corresponding problem of optimal fluctuational escape: Pontryagin’s Hamiltonian [20] in control theory can be identified with the Wentzel-Freidlin Hamiltonian [5] of the theory of fluctuations, and the optimal control force can be identified with one of the momenta of the Hamiltonian system [14,15,19]. It was therefore suggested that the optimal control function  $f_c(t)$  or  $f_i$  can be found via statistical analysis of the optimal fluctuational force [14,15,19,21–23]. In this technique the control functions  $f_c(t)$  and  $f_i$  in (1) and (2) are replaced by additive white Gaussian noise and the dynamics of the system is followed continuously. Several dynamical variables of the system and the random force are recorded simultaneously, and the statistics of all actual trajectories along which the system moves in a particular subspace of the coordinate space are then analyzed [24–26]. The *prehistory probability distribution* of trajectories moving the system from the equilibrium state to the remote state is sharply peaked about the optimal fluctuational path, thereby providing a solution to the control problem. We

note that this technique provides an experimental solution of the problem, which is especially useful from the point of view of applications.

For *continuous control*, it is useful to change variables  $\epsilon = v^{-1/2}$ ,  $\tau = \epsilon^{-1}t$ ,  $\Omega = \epsilon\omega$ ,  $z = \epsilon^{-1}(y - 1)$ , and  $x = \ln u$ . Following Pontryagin’s theory of optimal control, we then reduce the energy-minimal migration task to boundary problems for the Hamilton equation (cf. [14,15,19]).

$$\begin{aligned} \dot{x} &= z, & \dot{z} &= q - 1 + k \cos(\Omega\tau) - e^x(1 + \epsilon z) - \epsilon z + p_2, \\ \dot{p}_1 &= p_2 e^x(1 + \epsilon z), & \dot{p}_2 &= -p_1 + p_2 \epsilon(1 + e^x), \end{aligned} \quad (4)$$

with the boundary conditions [27]

$$\begin{aligned} (x(\tau_s), z(\tau_s), p_1(\tau_s), p_2(\tau_s)) &\in \mu^u, \\ (x(\tau_e), z(\tau_e), p_1(\tau_e), p_2(\tau_e)) &\in \mu^s, \end{aligned} \quad (5)$$

where  $\mu^u$  is an unstable manifold of  $C_3$  and  $\mu^s$  is a stable manifold of  $S_3$ ;  $\tau_s$  and  $\tau_f$  are initial and final times, respectively. Thus the boundary conditions (5) specify a heteroclinic trajectory of the system (4) (see [28,29] for details). The variable  $p_2$  gives the control function  $f$ .

The solution of the boundary problem (4) and (5) for the transition  $C_3 \rightarrow S_3$  was found by the shooting method starting from a guess derived from the prehistory approach [30]. The corresponding solution  $[x(\tau), p_2(\tau)]$  is shown in Figs. 1(b) and 1(c). Numerical simulations confirm that the control function  $f(\tau) = p_2(\tau)$  thus obtained does indeed induce migration from the cycle  $C_3$  to the cycle  $S_3$  in the optimal regime. Similar results are obtained for transition  $C_2 \rightarrow S_2$ .

For a *discrete time system*, the Pontryagin theory of optimal control can be extended to obtain an area-preserving map:

$$\begin{aligned} c_{i+1} &= q + G(c_i, \psi_i) e^{-T} + K \cos(\omega T + \psi_i) + p_{i+1}^c, \\ \varphi_{i+1} &= \varphi_i + \omega T, \quad \text{mod } 2\pi, \\ \begin{pmatrix} p_{i+1}^c \\ p_{i+1}^\varphi \end{pmatrix} &= \begin{pmatrix} \frac{\partial c_{i+1}}{\partial c_i} & \frac{\partial c_{i+1}}{\partial \varphi_i} \\ \frac{\partial \varphi_{i+1}}{\partial c_i} & \frac{\partial \varphi_{i+1}}{\partial \varphi_i} \end{pmatrix}^{-1} \begin{pmatrix} p_i^c \\ p_i^\varphi \end{pmatrix}, \end{aligned} \quad (6)$$

with the boundary conditions

$$(c_s, \varphi_s, p_s^c, p_s^\varphi) \in \mu^s, \quad (c_e, \varphi_e, p_e^c, p_e^\varphi) \in \mu^u, \quad (7)$$

where  $s$  and  $e$  are the initial and final instants of time, and  $\mu^u$  and  $\mu^s$  are an unstable manifold of  $C_3$  and a stable manifold of  $S_3$ , respectively;  $p_i^c$  determines the control function  $f_i$ . The corresponding boundary value problem (6) and (7) for the transition  $C_3 \rightarrow S_3$  can be solved by either the prehistory approach or a shooting method. Because of the complexity of the map, however, the accuracy of each method is limited and allows us to identify only a nearly optimal pulsed control function. The results of such an analysis for  $C_3 \rightarrow S_3$  in (2) are shown in Fig. 2. The prehistory approach gives somewhat better results, the corresponding energy  $J_d^{\text{stat}} \approx 3.58 \times 10^{-6}$  being less than that found by the shooting method,  $J_d^{\text{shoot}} \approx 3.7 \times 10^{-6}$

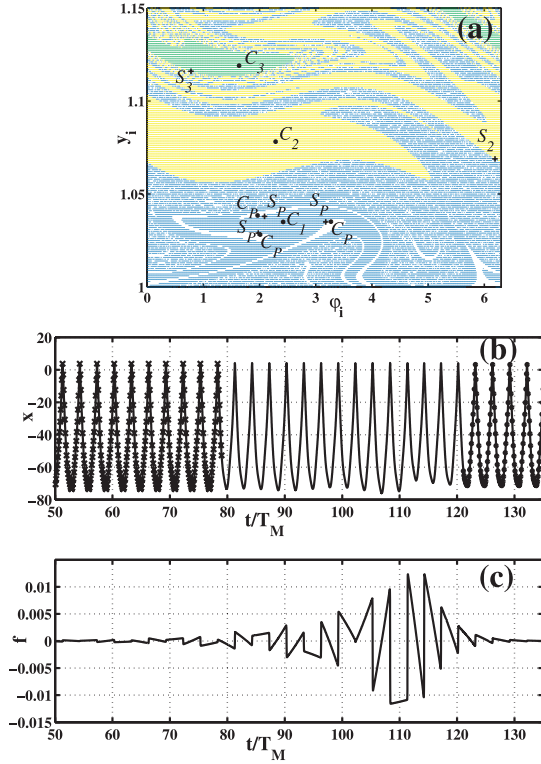


FIG. 1 (color). (a) Basins of attraction for the flow system (1), indicating some stable cycles  $C$  and saddle cycles  $S$ . Different colors correspond to the basins of different cycles. The cycles  $C_1$ ,  $C_2$ ,  $C_3$  and  $S_2$ ,  $S_3$  correspond to one pulse periodic motion, i.e., a fixed point in the Poincaré section; the cycles  $C_P$  and  $S_P$  correspond to three pulses periodic motion, i.e., a period-3 fixed point. Time realizations of the coordinate  $x(t)$  (b) and control force  $f(t)$  (c) are shown during migration from  $C_3$  to  $S_3$ . The stable cycle  $C_3$  and saddle cycle  $S_3$  are marked by  $\times$  and  $\bullet$ , respectively.

(and twice smaller than that of a single control impulse  $J_d^{\text{single}} \approx 8.4 \times 10^{-6}$ ). Similar results are obtained from an analysis of the optimal transition  $C_2 \rightarrow S_2$ . As a next step, we define (see above) a value of  $A$  for each type of pulsed control function, in order to apply the map results to the continuous system (1).

We have calculated directly the dependence of the energy (3) of the pulsed control function  $f_d(t)$  on the pulse duration  $\tau_c$ , finding that there is a minimum, i.e., an optimal duration  $\tau_c^{\text{opt}}$ , for which (3) is minimal. The optimal duration turns out to be less than the duration of a laser spike, so that (see above) the map (2) is applicable.

A *direct comparison* between the continuous and different pulsed methods of control is given in Table I. It shows that continuous control is energetically far more efficient than pulsed control. That can be explained by the short duration  $\tau$  of impulses of the pulsed force. In turn, a pulsed control function consisting of a sequence of impulses (multipulsed force) is more efficient than a single-pulse control function. It is evident that the energies of the pulsed control functions can be decreased by orders of magnitude

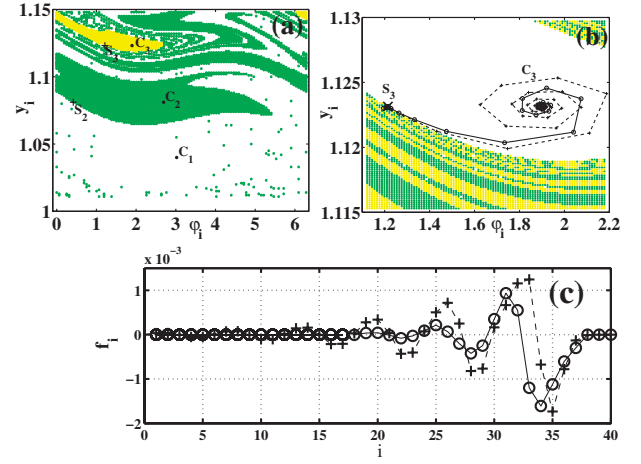


FIG. 2 (color). (a) Basins of attraction for the map (2), indicating some stable cycles  $C$  and saddle cycles  $S$ . Different colors correspond to the basins of different cycles. (b) Migration trajectory from  $C_3$  to  $S_3$  in the map (2), and (c) realizations of the corresponding control force. Dashed lines and markers  $+$  correspond to solutions of the boundary problem, and the solid line with  $\circ$  indicates the solution based on fluctuational prehistory analysis.

by optimization of the impulse duration. Even so, the optimized energy of the pulsed control function still exceeds the energy of the continuous force by 2 orders of magnitude.

We have also investigated the influence of the phase relationship between the single control impulse and the laser oscillations, i.e., the different Poincaré sections when the single control impulse acts [30]. We found that (i) the control energy  $J_c^i$  changes with phase by a factor of up to  $2\times$  and (ii) the map (2) is close to the optimal phase relationship (cf. values of  $J_c^{\text{in}}$  and  $J_c^{\text{opt}}$  in Table I).

Although the continuous and discrete laser models describe the system dynamics well on long time scales and yield quantitatively similar basins of attractions for the stable limit cycles [cf. Figs. 1(a) and 2(a)], estimates of stability and of the energies of optimal control functions may differ by a large factor. A possible reason lies in the particular form of control function  $f_i$  in the map (2): we choose a force that is additive in the map. So we might expect a different conclusion for another form of control function  $f_i$ . The latter can be identified by detailed inspection of the continuous function  $f(t)$  in Fig. 1(c), which has a complex structure; but it can be emulated by a sequence of impulses of duration close to the period of the cycle  $C_3$  or to the time interval  $3T_M$  between laser spikes. We also note that external driving changes the value of the pump, i.e., of  $q$ . Hence we can suggest a new form of control for the map:

$$\begin{aligned} c_{i+1} &= q_i + G(c_i, \psi_i, q_i)e^{-T} + K \cos(\omega T + \psi_i), \\ \varphi_{i+1} &= \varphi_i + \omega T, \text{ mod } 2\pi, \quad q_{i+1} = q_i + f_i, \end{aligned} \quad (8)$$

where  $q$  remains constant between Poincaré sections and

TABLE I. The energy of the optimal control function for controlling migration between cycles in the continuous system (1), expressed in dimensionless units.  $J_c$  corresponds to the continuous function obtained by solution of the boundary problem (5) for the system (4). The energy  $J_c^p$  corresponds to the multipulsed control functions determined by the prehistory approach, whereas  $J_c^i$  corresponds to a single impulse function. The energies  $J_c^p$  and  $J_c^i$  were determined for very short-duration control impulses  $\tau_c \ll T_M$ , approximating  $\delta$  functions, where  $T_M$  is the period of the external pumping. The energies  $J_c^{p\text{opt}}$  and  $J_c^{i\text{opt}}$  correspond to the multipulsed and single-pulse control functions, respectively, with an optimal duration  $\tau_c^{\text{opt}}$ . The energies  $J_c^{\text{pn}}$  and  $J_c^{\text{in}}$  were determined for the map (8) and correspond to a multipulsed and single-pulse control functions, respectively. The value  $J_{\text{opt}}^i$  was obtained by double optimization of the duration of the single control impulse and of the location of Poincaré section.

Transition	$J_c$	$J_c^p/J_c^{p\text{opt}}$	$J_c^i/J_c^{i\text{opt}}$	$J_c^{\text{pn}}$	$J_c^{\text{in}}/J_{\text{opt}}^i$
$C_3 \rightarrow S_3$	0.00004	0.08275/0.00275	0.16/0.005	0.000077	0.0001/0.000098

changes at each cross section. For (8) we formulate the same boundary problem as for (2), and we use the same method to determine the control function. The results of a prehistory analysis and for a single impulse are presented in two last columns of Table I. The energy of the control function is significantly decreased, but it is still nearly twice as large as that of the continuous function.

Summarizing, we have found the energy-optimal control function for effecting migration of a class-B laser from its stable limit cycle to a saddle cycle, for both continuous and discrete descriptions of the laser. This allows us to compare the efficiency of the two techniques directly, for the first time, showing that continuous control is the more energy efficient and provides more accurate estimates of the stability of quasistable states. The fact that the optimal form of continuous force [Fig. 1(c)] is closer to a sequence of impulses than to a harmonic force allows us to suggest a pulsed control function approaching continuous efficiency. We note that specific targeting of periodic orbits has been achieved experimentally by single-shot perturbation of intracavity losses in a CO<sub>2</sub> laser [31,32] that can be described by the continuous and discrete laser equations used in this Letter. The results obtained can be verified directly in experiment and therefore applied, e.g., to phase coding information schemes.

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