Bosonization, Pairing, and Superconductivity of the Fermionic Tonks-Girardeau Gas

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We determine some exact static and time-dependent properties of the fermionic Tonks-Girardeau (FTG) gas, a spin-aligned one-dimensional Fermi gas with infinitely strongly attractive zero-range odd-wave interactions. We show that its two-particle reduced density matrix exhibits superconductive off-diagonal long-range order, and on a ring an FTG gas with an even number of atoms has a highly degenerate ground state with quantization of Coriolis rotational flux and high sensitivity to rotation and to external fields and accelerations. For a gas initially under harmonic confinement, we show that during an expansion the momentum distribution undergoes a "dynamical bosonization," approaching that of an ideal Bose gas without violating the Pauli exclusion principle.

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If an ultracold atomic vapor is confined in a de Broglie waveguide with transverse trapping so tight and temperature so low that the transverse vibrational excitation quantum $\hbar\omega$ is larger than available longitudinal zero point and thermal energies, the effective dynamics becomes one-dimensional (1D) [1,2], a regime currently under intense experimental investigation [3,4]. Confinement-induced 1D Feshbach resonances (CIRs) reachable by tuning the 1D coupling constant via 3D Feshbach scattering resonances occur for both Bose gases [1] and spin-aligned Fermi gases [5]. Near a CIR, the 1D interaction is very strong, leading to strong short-range correlations, breakdown of effective-field theories, and emergence of highly correlated N-body ground states. In the bosonic case with very strong repulsion [1D hard-core Bose gas with coupling constant $g_{1D}^B \to +\infty$, the Tonks-Girardeau (TG) gas], the exact N-body ground state was determined some 45 years ago by a Fermi-Bose (FB) mapping to an ideal Fermi gas [6], leading to "fermionization" of many properties of this Bose system, as recently confirmed experimentally [4]. The "fermionic TG" (FTG) gas [7], a spin-aligned Fermi gas with very strong *attractive* 1D odd-wave interactions, can be realized by 3D Feshbach resonance mediated tuning to the attractive side of the CIR with 1D coupling constant $g_{1D}^F \to -\infty$. It has been pointed out [5,7] that the generalized FB mapping [5,7,8] can be exploited in the opposite direction to map this system to the trapped *ideal Bose* gas, leading to determination of the exact N-body ground state and "bosonization" of many properties of this Fermi system. We recently examined the equilibrium one-body density matrix and exact dynamics following sudden turnoff of the interactions by detuning from the CIR [9]. Here we determine some other exact properties of the untrapped, ring-trapped, and harmonically trapped fermionic TG gas, the most striking of which are pairing, superconductive off-diagonal long-range order (ODLRO) of the two-body density matrix, a highly degenerate ground state of an even number of atoms on a ring with quantization of Coriolis rotational flux and high sensitivity to rotation and to external fields and accelerations, and a "dynamical bosonization" of the momentum distribution following sudden relaxation of the trap frequency.

Untrapped FTG gas.—The Hamiltonian is $\hat{H} = \sum_{j=1}^{N}[-(\hbar^2/2m)(\partial^2/\partial x_j^2)] + \sum_{1\le j<\ell\le N} v_{\rm int}^F(x_j - x_\ell)$, where $v_{\rm int}^F$ is the two-body interaction. Since the spatial wave function is antisymmetric due to spin polarization, there is no s-wave interaction, but it has been shown [5,7] that a strong, attractive, short-range odd-wave interaction (1D analog of 3D p-wave interactions) occurs near the CIR. This can be modeled by a narrow and deep square well of depth V_0 and width $2x_0$. The contact condition at the edges of the well is [7] $\psi_F(x_{j\ell} = x_0) = -\psi_F(x_{j\ell} = -x_0) = -a_{1D}^F \psi_F'(x_{j\ell} = \pm x_0)$, where a_{1D}^F is the 1D scattering length and the prime denotes differentiation. Consider first the relative wave function $\psi_F(x)$ in the case $N = 2$. The FTG limit is $a_{1D}^F \to -\infty$, a zero-energy scattering resonance. The exterior solution is $\psi_F(x) = \mathrm{sgn}(x) = \pm 1$ ($+1$ for $x > 0$ and -1 for $x < 0$), and the interior solution fitting smoothly onto this is $\sin(\kappa x)$, with $\kappa = \sqrt{mV_0/\hbar^2} = \pi/2x_0$. In the zero-range limit $x_0 \to 0+$, the well area $2x_0V_0 = (\pi\hbar)^2/2mx_0 \to \infty$, stronger than a negative delta function. In this limit, the wave function is discontinuous at contact $x_0 = 0\pm$, allowing an infinitely strong zero-range interaction in spite of the antisymmetry of ψ_F [8]. This generalizes immediately to arbitrary N: The exact FTG gas ground state is

$$\psi_F(x_1, \ldots, x_N) = A(x_1, \ldots, x_N) \prod_{j=1}^{N} \phi_0(x_j), \qquad (1)$$

with $A(x_1, \ldots, x_N) = \prod_{1\le j<\ell\le N} \mathrm{sgn}(x_\ell - x_j)$ the "unit antisymmetric function" employed in the original discovery of fermionization [6] and $\phi_0 = 1/\sqrt{L}$ the ideal Bose gas ground orbital, L being the periodicity length. Its energy is zero [10] and it satisfies periodic boundary conditions for odd N and antiperiodic boundary conditions for even N [11].

The exact single-particle density matrix $\rho_1(x,x') = N\int \psi_F(x,x_2,\ldots,x_N)\psi_F^*(x',x_2,\ldots,x_N)dx_2,\ldots,dx_N$ is [9,12] $\rho_1(x,x') = N\phi_0(x)\phi_0^*(x')[F(x,x')]^{N-1}$, with $F(x,x') = \int_{-L/2}^{L/2} \mathrm{sgn}(x-y)\mathrm{sgn}(x'-y)|\phi_0(y)|^2 dy = 1 - 2|x-x'|/L$. In the thermodynamic limit $N\to\infty$, $L\to\infty$, $N/L = n$, this gives an exponential decay [12]: $\rho_1(x,x') = ne^{-2n|x-x'|}$. Its Fourier transform n_k, normalized to $\sum_k n_k = N$ (allowed momenta $\nu 2\pi/L$, with $\nu = 0, \pm 1, \pm 2, \ldots$), is the momentum distribution function $n_k = [1 + (k/2n)^2]^{-1}$. It satisfies the exclusion principle limitation $n_k \le 1$, but nevertheless, for $n\to 0$ the continuous momentum density $n(k) = (L/2\pi)n_k$ reduces to N times a representation of the Dirac delta function, simulating the ideal *Bose* gas distribution: $n(k) \longrightarrow_{n\to 0} N\delta(k)$ [12].

The two-particle density matrix $\rho_2(x_1,x_2;x_1',x_2') = N(N-1)\int \psi_F(x_1,\ldots,x_N)\psi_F^*(x_1',x_2',x_3,\ldots,x_N) \times dx_3,\ldots,dx_N$ also has a simple closed form:

$$\begin{aligned}\rho_2(x_1,x_2;x_1',x_2') = N(N-1)&\mathrm{sgn}(x_1 \\ &- x_2)\phi_0(x_1)\phi_0(x_2)\mathrm{sgn}(x_1' \\ &- x_2')\phi_0^*(x_1')\phi_0^*(x_2') \\ &\times [G(x_1,x_2;x_1',x_2')]^{N-2}, \end{aligned} \qquad (2)$$

where $[G(x_1,\, x_2;\, x_1',\, x_2')]^{N-2} = [\int_{-L/2}^{L/2} \mathrm{sgn}(x_1 - x) \times \mathrm{sgn}(x_2 - x)\mathrm{sgn}(x_1' - x)\mathrm{sgn}(x_2' - x)|\phi_0|^2(x)dx]^{N-2} = e^{2n(y_1 - y_2 + y_3 - y_4)}$ in the thermodynamic limit, and $y_1 \le y_2 \le y_3 \le y_4$ are the arguments $(x_1, x_2; x_1', x_2')$ in ascending order. ρ_2 is of order n^2 in the following cases: (a) $|x_1 - x_1'| \le O(1/n)$, $|x_2 - x_2'| \le O(1/n)$; (b) $|x_1 - x_2'| \le O(1/n)$, $|x_2 - x_1'| \le O(1/n)$; (c) $|x_1 - x_2| \le O(1/n)$, $|x_1' - x_2'| \le O(1/n)$. These are just Yang's criteria [13] for superconductive ODLRO of ρ_2 in the absence of ODLRO of ρ_1. In case (c), ρ_2 remains of order n^2 for arbitrarily large separation of the centers of mass $X = (x_1 + x_2)/2$ and $X' = (x_1' + x_2')/2$, the hallmark of ODLRO. On the other hand, in cases (a) and (b), ρ_2 decays exponentially with $|X - X'|$. In the thermodynamic limit, only configurations (c) contribute to the largest eigenvalue of ρ_2, and ρ_2 separates apart from negligible contributions (a) and (b) [14]:

$$\begin{aligned}\rho_2(x_1,x_2;x_1',x_2') = n^2\,&\mathrm{sgn}(x_1 - x_2)e^{-2n|x_1-x_2|}\mathrm{sgn}(x_1' - x_2') \\ &\times e^{-2n|x_1'-x_2'|} + \text{terms negligible for } \lambda_1.\end{aligned} \qquad (3)$$

By Yang's argument [13], the largest eigenvalue λ_1 is of order N, and this is confirmed by comparison with the λ_1 contribution $\lambda_1 u_1(x_1,x_2)u_1(x_1',x_2')$ to the spectral representation of ρ_2, implying that the corresponding eigenfunction is $u_1(x_1,x_2) = \mathcal{C}\,\mathrm{sgn}(x_1 - x_2)e^{-2n|x_1-x_2|}$, with [15] $\mathcal{C} = \sqrt{2n/L}$, implying $\lambda_1 = n^2/\mathcal{C}^2 = N/2$. The range $1/2n$ of u_1 is in the region of onset of a Bose-Einstein condensation (BEC)-BCS crossover between tightly bound bosons and loosely bound Cooper pairs. There is an upper bound [13] $\lambda_1 \le N$ on the largest eigenvalue, so the FTG gas is highly superconductive in the sense of Yang's ODLRO criterion.

FTG gas on a ring.—If the FTG gas is trapped on a circular loop of radius R, with particle coordinates x_j measured around the circumference $L = 2\pi R$, the FTG gas must satisfy periodic boundary conditions for both odd and even N because of the single-valuedness of its wave function. Since the mapping function $A(x_1,\ldots,x_N) = \prod_{1\le j<\ell\le N}\mathrm{sgn}(x_\ell - x_j)$ is periodic (antiperiodic) for odd (even) N as a result of its definition, it follows that the mapped ideal Bose gas used to solve the FTG problem must satisfy periodic (antiperiodic) boundary conditions for odd (even) N. The ground state of a FTG gas on a ring is then different depending on the particle number parity. For odd N, the FTG ground state in Eq. (1) is built from the zero-momentum orbital $\phi_0 = 1/\sqrt{L}$ and corresponds to mapping the FTG gas onto the ideal Bose gas ground state, the usual complete BEC, and is nondegenerate. On the other hand, for even N, which we henceforth assume, antiperiodicity requires that the only plane-wave orbitals allowed are $e^{ikx_j}/\sqrt{L}$, with $k = \pm\pi/L, \pm 3\pi/L, \ldots$. The ground state of this fictitious ideal Bose gas, and hence that of the mapped FTG gas, is then $(N+1)$-fold degenerate, with energy eigenvalue $N(\hbar^2/2m)(\pi/L)^2$. These degenerate ground states are fragmented BECs, with wN atoms in the orbital $e^{i\pi x_j/L}$ and $(1-w)N$ in $e^{-i\pi x_j/L}$ with $0 \le w \le 1$, and are conveniently labeled by a quantum number $\ell_z = (w - \frac{1}{2})N = 0, \pm 1, \pm 2, \ldots, \pm\frac{N}{2}$ related to the eigenvalue P of the circumferential linear momentum and that L_z of the angular momentum z component by $P = \ell_z\hbar/R$ and $L_z = \ell_z\hbar$. The angular momentum per particle is half-integral due to antiperiodicity of the orbitals, and the degenerate ground states are in 1–1 correspondence with the eigenstates of the *spin* angular momentum z component of N spin-$1/2$ fermions.

The ground state degeneracy makes the FTG gas on a ring a good candidate for detecting small external fields and linear accelerations. Suppose that there is a potential gradient parallel to a diameter of the ring or an acceleration leading to a gradient in the inertial potential arising from Einstein's principle of equivalence, with the circumferential minimum of this potential occurring at a point x_0. Then the degeneracy is lifted, and, to lowest order in degenerate perturbation theory, all N atoms occupy the orbital $\phi_0(x) = \sqrt{2/L}\cos[\pi(x - x_0)/L]$, leading to an observable asymmetric density profile $n(x) = 2n\cos^2[\pi(x - x_0)/L]$.

Because of its quantum coherence, the FTG gas is also a good candidate for a sensitive rotation detector. Suppose that the ring trap is rotating with angular velocity $\vec{\omega}$ perpendicular to the plane of the ring. In the rotating coordinate system, each atom sees an effective Coriolis force $\vec{F}_{\mathrm{Cor}} = 2m\vec{v}\times\vec{\omega}$. Comparing this with the usual magnetic force $\vec{F}_{\mathrm{mag}} = (e/c)\vec{v}\times\vec{B}$, one sees that the kinetic energy operators in the Hamiltonian in the rotating system are $[\hat{p}_j - (h/L)(\Phi/\Phi_0)]^2/2m$, where $\hat{p}_j = (\hbar/i)\partial/\partial x_j$, $\Phi = \pi R^2 \omega$ is the Coriolis flux through the loop, and $\Phi_0 = h/2m$ is the Coriolis flux quantum. The energy of each state $|\ell_z\rangle$ then becomes $E = E_0(\Phi = 0) + (N\hbar^2/2mR^2)[(\Phi/\Phi_0)^2 - 2\ell_z(\Phi/\Phi_0)]$, which is minimized when $\ell_z = \frac{N}{2}$ if $\Phi > 0$ and $\ell_z = -\frac{N}{2}$ if $\Phi < 0$; i.e., even a very small angular velocity leads to a nondegenerate ground state with all N atoms at either $k = \pi/L$ or $k = -\pi/L$. Generalizing to states differing from the $\Phi = 0$ ground states by displacement in k space by integral multiples of $2\pi/L$, one obtains the Φ-dependent ground state energy $E_0(\Phi)$ shown by the heavy line in Fig. 1, in which the lighter lines show the lowest energies for $\ell_z = \pm\frac{N}{2}, \pm\frac{3N}{2}, \ldots$. The ground state energy is a periodic function of Φ with period Φ_0 in accord with a general theorem [13], but, unlike the usual situation for a superconductor, (a) there is no smaller period $\Phi_0/2$, and (b) for even N, $\Phi_0 = 0$ is a relative maximum of E_0 rather than a minimum (as is the case of odd N), the first minima occurring at $\Phi = \pm\Phi_0/2$. The barrier heights of the energy landscape in Fig. 1 vanish like $1/N$ for $N \to \infty$, so flux quantization will not be observable for a macroscopic ring. However, it may be observable for mesoscopic rings using BEC-on-a-chip technology. For example, assuming a ring radius $R = 5\ \mu$m, one finds that, for ^{6}Li, $\Delta E > k_B T$ for $T < 50$ nK.

Flow properties on a nonrotating ring.—According to the FB mapping the excitation spectrum of the FTG gas is the same as that of an ideal Bose gas, and, hence, it is sufficient to analyze the latter. Since the excitation energy of the ideal Bose gas is quadratic in the excitation momentum $\hbar q$, the FTG gas does not satisfy the Landau-Bogoliubov criterion for superfluidity. We investigate here the possibility of flow metastability associated with barriers in the excitation energy landscape as a function of the transferred momentum. It was shown by Bloch [16] that for the usual ideal Bose gas, which corresponds to the case of odd N in our treatment, no such barriers exist. In the case of even N, both the ground state and the excitation branches are $(N+1)$-fold degenerate, but it is sufficient here to consider the $\ell_z = 0$ ground state and the excitations arising from it by promoting atoms to higher k values. Generalizing Bloch's analysis, we note that for $0 < \nu \leq N/2$ the lowest branch corresponds to excitation of ν atoms from $k = -\pi/L$ to $k = 3\pi/L$, yielding a state with angular momentum z component $\ell_z\hbar$, with $\ell_z = 2\nu$, and with excitation energy $\epsilon(\ell_z) = \ell_z\hbar^2/mR^2$. At $\nu = N/2$, one has reached a state differing from the ground state by translation of all atoms by an amount $2\pi/L$ in k space, and one can repeat this process, promoting atoms from $k = \pi/L$ to $5\pi/L$, yielding another straight-line segment connecting the points $\ell_z = N$ and $\ell_z = 2N$ on a parabolic curve $(\ell_z\hbar)^2/2NmR^2$, etc. Together with symmetry $\epsilon(\ell_z) = \epsilon(-\ell_z)$, this yields an excitation energy curve composed of straight-line segments as in the dashed curve of Bloch's Fig. 2 [16] with the notation $P = \ell_z\hbar/R$. Hence, for both odd and even N there are no energy barriers, and the FTG gas on a nonrotating ring does not exhibit flow metastability.

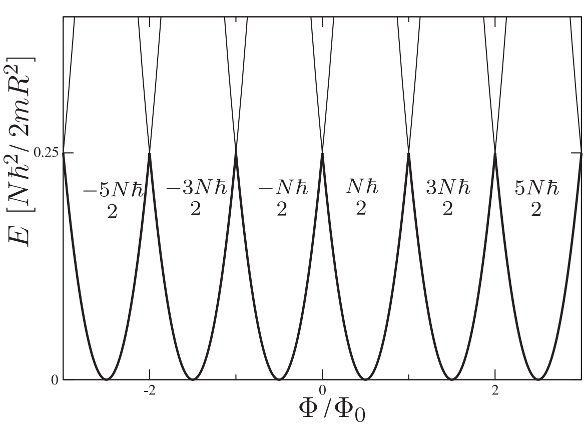

FIG. 1. Dependence of energies E on rotational flux Φ. Heavy line: Ground state energy $E_0(\Phi)$. Lighter lines: Lowest energy for each value of total angular momentum.

Expansion from a longitudinal harmonic trap.—We focus finally on a 1D expansion, as could be achieved by keeping on the transverse confinement. If the 1D interactions are suddenly turned off before the gas is let free to expand from a longitudinal harmonic trap, the density profile at long times reflects the initial momentum distribution [9]. If, instead, the interactions are kept on during the expansion, we find that the density profile expands self-similarly, while the momentum distribution evolves from an initial overall Lorentzian shape [12] to that of an ideal Bose gas. These properties can be demonstrated with the aid of an exact scaling transformation as we outline below. Since the FB mapping holds also for time-dependent phenomena induced by one-body external fields [17], the exact many-body wave function $\psi_F(x_1, \ldots, x_N; t) = A(x_1, \ldots, x_N)\prod_{j=1}^{N}\phi_0(x_j; t)$ during the dynamics is fully determined by the solution of the single-particle Schrödinger equation for the orbital $\phi_0(x_j; t)$. For the case of an external potential $V_{\rm ext}(x, t) = m\omega(t)^2x^2/2$, with $\omega(0) = \omega_0$, the solution is known [18] to be $\phi_0(x; t) = \phi_0(x/b(t); 0)e^{imx^2\dot{b}/2b\hbar - iE_0\tau(t)/\hbar}$, where $b(t)$ is the solution of the differential equation $\ddot{b} + \omega^2(t)b = \omega_0^2/b^3$, with $b(0) = 1$ and $\dot{b}(0) = 0$, $\tau(t) = \int_0^t dt' 1/b^2$, and $E_0 = \hbar\omega_0/2$. Since the unit antisymmetric wave function A is invariant under the scaling transformation, we immediately obtain the expression for the many-body wave function, $\psi_F(x_1, \ldots, x_N; t) = b^{-N/2}\psi_F(x_1/b, \ldots, x_N/b; 0)e^{i(\dot{b}/b\omega_0)\sum_{j=1}^{N}x_j^2/2x_{\rm osc}^2}e^{-iNE_0\tau(t)/\hbar}$, and for the one-body density matrix, $\rho_1(x, x'; t) = \frac{1}{b}\rho_1(\frac{x}{b}, \frac{x}{b}; 0)\exp\{-i(\dot{b}/b)[(x^2 - x'^2)/2x_{\rm osc}^2]\}$. This yields the momentum distribution as a function of time. While the intermediate-time dynamics has to be determined numerically, the stationary-phase method determines the long-time evolution of the momentum distribution in the

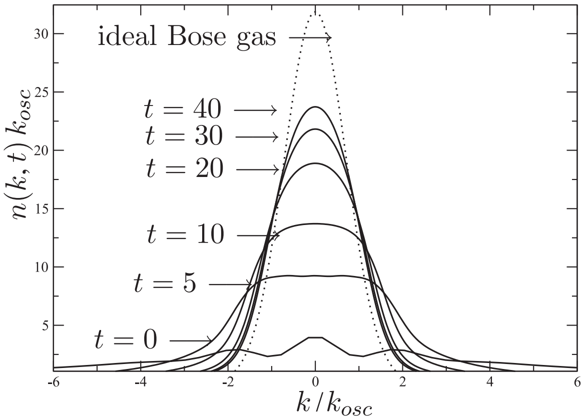

FIG. 2. Momentum distributions of a FTG gas (solid lines) with $N = 9$ particles as functions of the wave vector k at subsequent times t (in units of $1/\omega_0$) during a 1D expansion and asymptotic long-time expression (4) (dashed line).

same way as for the bosonic TG gas [19]. For the case of a 1D expansion, the scaling parameter is $b(t) = \sqrt{1 + \omega_0^2 t^2}$, and the momentum distribution tends to that of an ideal Bose gas under harmonic confinement,

$$n(k, t \to \infty) \simeq |\omega_0/\dot{b}| n_B(k\omega_0/\dot{b}), \tag{4}$$

where $n_B(k) = 2\pi N |\tilde{\phi}_0(k)|^2$, with $\tilde{\phi}_0(k) = \pi^{-1/4} k_{\rm osc}^{-1/2} e^{-k^2/2k_{\rm osc}^2}$ and $k_{\rm osc} = 1/x_{\rm osc}$. This behavior is illustrated in Fig. 2. Quite noticeably, the bosonization time appears to be much longer than the fermionization time of the momentum distribution of the bosonic TG gas [19]. Note that the "dynamical bosonization" described above does not violate the Pauli exclusion principle: By using the above scaling solution for the one-body density matrix and fixing unit normalization of the natural orbitals at all times, it follows that the eigenvalues α_j of $\rho_1(x, x'; t)$ are invariant during the expansion and, hence, always satisfy the condition $\alpha_j \leq 1$.

In conclusion, we have found that (a) the untrapped system exhibits superconductive ODLRO of the two-body density matrix ρ_2 associated with its maximal eigenvalue $N/2$ and pair eigenfunction $\mathcal{C}\,\mathrm{sgn}(x_1 - x_2) \times e^{-2n|x_1 - x_2|}$; (b) on a ring it has a highly degenerate ground state for an even atom number, and it exhibits quantization of rotational Coriolis flux and high sensitivity to rotation and to accelerations, making it a good candidate for high-sensitivity detectors; (c) the harmonically trapped system undergoes a "dynamical bosonization" of its momentum distribution during a 1D expansion.

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