

## Casimir Interaction between a Plate and a Cylinder

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We find the exact Casimir force between a plate and a cylinder, a geometry intermediate between parallel plates, where the force is known exactly, and the plate sphere, where it is known at large separations. The force has an unexpectedly weak decay  $\sim L/[H^3 \ln(H/R)]$  at large plate-cylinder separations  $H$  ( $L$  and  $R$  are the cylinder length and radius), due to transverse magnetic modes. Path integral quantization with a partial wave expansion additionally gives a qualitative difference for the density of states of electric and magnetic modes, and corrections at finite temperatures.

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With recent advances in the fabrication of electronic and mechanical systems on the nanometer scale, quantum effects like Casimir forces have become increasingly important [1,2]. These systems can probe mechanical oscillation modes of quasi-one-dimensional structures such as nanowires or carbon nanotubes with high precision [3]. However, thorough theoretical investigations of Casimir forces are to date limited to “closed” geometries such as parallel plates [4] or, recently, a rectilinear “piston” [5], where the zero point fluctuations are not diffracted into regions which are inaccessible to classical rays. A notable exception is the original work by Casimir and Polder on the interaction between a plate and an atom (sphere) at asymptotically large separation [6].

In this Letter we consider the electrodynamic Casimir interaction between a plate and a parallel cylinder (or “wire”), both assumed to be perfect metals (see inset of Fig. 1). We show that the Casimir interaction can be computed without approximation for this geometry. We believe that the methods presented here may yield exact solutions for other interesting geometries as well. This geometry is also of recent experimental interest: keeping two plates parallel has proved very difficult. The sphere and plate configuration avoids this problem, but the force is not extensive. The cylinder is easier to hold parallel and the force is extensive in its length [7].

Casimir interactions, while attractive for perfect metals in all known cases, depend strongly on geometry. Consider the Casimir interaction energy (discarding separation independent terms) at asymptotically large  $H$  for three fundamental geometries which differ in the codimension of the surfaces [8]: two plates, plate-cylinder, and finally, plate-sphere, corresponding to codimension 1, 2, and 3, respectively. It is instructive to consider both a scalar field which vanishes on the surfaces ( $D \equiv$  Dirichlet) and the electromagnetic field (EM). For parallel plates (area  $A$ )  $E \sim \hbar c A/H^3$  in both cases [4]. For a plate and a sphere of radius  $R$ ,  $E \sim \hbar c R/H^2$  [9] for the Dirichlet case, as compared with  $E \sim \hbar c R^3/H^4$  for the EM case [6]. Based on these results, expectations for the plate and cylinder

geometry might range from  $\sim \hbar c L R^2/H^4$ , proportional to the cylinder volume, to  $\sim \hbar c L R/H^3$ , proportional to its surface area, or even  $\sim \hbar c L/H^2$  with a potential nonpower law dependence on the radius.

A simple but uncontrolled method for study of non-planar geometries is the proximity force approximation (PFA), where the system is treated as a sum of infinitesimal parallel plates [10]. Applied to the plate-cylinder geometry, the PFA yields  $E_{\text{PFA}} = -\frac{1}{960} \pi^3 \hbar c L \sqrt{R/2a^5}$  to leading order in  $a/R$ , where  $a = H - R$ . Other approximations include semiclassical methods based on the Gutzwiller trace formula [11], and a recent optical approach which sums also over closed but nonperiodic paths [12]. For large

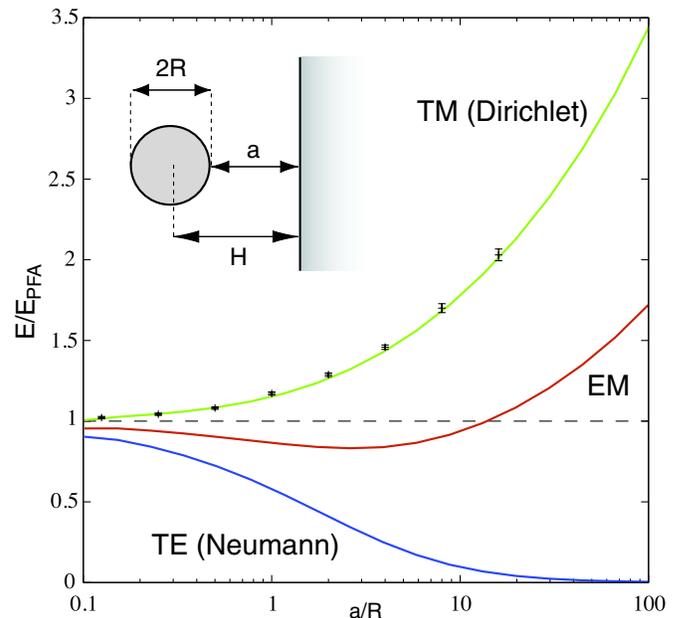


FIG. 1 (color online). Ratio of the exact Casimir energy to the PFA for the geometry shown in the inset. All curves are obtained at order 25 of the partial wave expansion, and the accuracy lies within the line thickness even at small  $a/R$ . The Dirichlet data points are from Ref. [19].

separations, a multiple scattering approach is available [13], but has not been adapted to this geometry. For the Dirichlet case, a Monte Carlo approach based on world line techniques has been applied to the plate-cylinder case [14].

Our result provides a test for the validity of these approximate schemes, and also provides insight into the large distance limit. In particular, we find the unexpected result that the electrodynamic Casimir force for the plate-cylinder geometry has the weakest of the possible decays,

$$F = -\frac{1}{8\pi} \frac{\hbar c L}{H^3 \ln(H/R)}, \quad (1)$$

as  $H/R \rightarrow \infty$ . The same asymptotic result applies to a scalar field with Dirichlet boundary conditions. Interestingly, the decay exponent of the force is *not monotonic* in the number of codimensions:  $(-4, -3 - \epsilon, -5)$  for codimension  $(1, 2, 3)$ , respectively. In contrast, the Dirichlet case is monotonic with exponents  $(-4, -3 - \epsilon, -3)$ .

In the remainder we derive these results, summarized in Eqs. (5)–(8), using path integral techniques. Our approach also yields the distance dependent part of the density of states, and thermal contributions to the force.

The translational symmetry along the cylinder axis enables a decomposition of the EM field into transverse magnetic (TM) and electric (TE) modes [15] which are described by a scalar field obeying Dirichlet (D) or Neumann (N) boundary conditions, respectively. We can compare our TM results to recent Monte Carlo Dirichlet results [14]. Moreover, the mode decomposition turns out to be useful also in identifying the physical mechanism behind the weak decay of the force, which at large distance is fully dominated by D modes.

Our starting point is a path integral representation [16] for the effective action which yields a trace formula for the density of states (DOS) [17]. The latter is then evaluated using a partial wave expansion. The DOS on the imaginary frequency axis is related to a Green's function by  $\rho(iq_0) = (2q_0/\pi) \int d^3\mathbf{x} G(\mathbf{x}, \mathbf{x}; q_0)$ , where  $G(\mathbf{x}, \mathbf{x}'; q_0)$  is the Green's function for the scalar field with action  $S = \frac{1}{2} \times \int d^3x (|\nabla\Phi|^2 + (q^0)^2\Phi^2)$ . The effect of boundaries on the Green function can be obtained by placing functional delta functions on the boundary surfaces in the functional integral [16]. By integrating out both the field  $\Phi$  and the auxiliary fields which represent the delta functions on the surfaces, one obtains the trace formula [17]

$$\delta\rho(q_0) = -\frac{1}{\pi} \frac{\partial}{\partial q_0} \text{Tr} \ln(MM_\infty^{-1}), \quad (2)$$

where  $\delta\rho(q_0)$  is the change in the DOS caused by moving the plate and cylinder in from infinity. The information about geometry is contained in the matrix  $M$  of the quadratic action for the auxiliary fields, given by  $M_{\alpha\beta}(\mathbf{u}, \mathbf{u}'; q_0) = G_0(\mathbf{s}_\alpha(\mathbf{u}) - \mathbf{s}_\beta(\mathbf{u}'); q_0)$  for D and by  $M_{\alpha\beta}(\mathbf{u}, \mathbf{u}'; q_0) = \partial_{\mathbf{n}_\alpha(\mathbf{u})} \partial_{\mathbf{n}_\beta(\mathbf{u}')} G_0(\mathbf{s}_\alpha(\mathbf{u}) - \mathbf{s}_\beta(\mathbf{u}'); q_0)$  for N boundary conditions;  $G_0 = e^{-q_0|\mathbf{x}|}/4\pi|\mathbf{x}|$  is the free space Green function,  $\partial_{\mathbf{n}_\alpha}$  is its derivative normal to the

surface, and  $\mathbf{s}_\alpha(\mathbf{u})$  parametrizes the surfaces (which are numbered by  $\alpha = 1, 2$ ) in terms of surface coordinates  $\mathbf{u}$ .  $M_\infty^{-1}$  is the functional inverse of  $M$  at infinite surface separation. The trace in Eq. (2) runs over  $\mathbf{u}$  and  $\alpha$ . For the cylinder with its axis oriented along the  $x_1$  direction we set  $\mathbf{s}_1(x_1, \varphi) = (x_1, R \sin(\varphi), R \cos(\varphi))$  and for the plate  $\mathbf{s}_2(x_1, x_2) = (x_1, x_2, H)$  (see inset of Fig. 1).

The Casimir energy of interaction is given by  $E = (\hbar c/2) \int_0^\infty dq_0 q_0 \delta\rho(q_0)$ . After transforming to momentum space,  $M$ , the Fourier transform of the matrix  $M$  has block diagonal form with respect to  $q_0$  and the momentum  $q_1$  along the cylinder axis, so the Casimir energy can be expressed as,

$$E = \frac{\hbar c L}{8\pi^2} \iint dq_0 dq_1 \ln \frac{\det \tilde{M}(q_0, q_1)}{\det \tilde{M}_\infty(q_0, q_1)}. \quad (3)$$

The elements of the matrix  $M$  are labeled by the integer index  $m = -\infty, \dots, \infty$  coming from the compact  $\varphi$  dimension of the cylinder, and the momentum  $q_2$  along the other direction parallel to the plate, to read

$$\tilde{M} = \begin{pmatrix} A_{[m, m']} & B_{[m, q_2']} \\ B_{[q_2, m']}^T & C_{[q_2, q_2']} \end{pmatrix}. \quad (4)$$

The matrix  $A_{[m, m']}$  is diagonal, with elements  $A_{[m, m]} \equiv A_m = I_m(ru)K_m(ru)$  for D and  $A_m = (u/H)^2 I_m'(ru) \times K_m'(ru)$  for N modes. The matrix  $C$  also has only diagonal elements  $C_{[q_2, q_2]} \equiv C(q_2) = H/(2\sqrt{u^2 + u_2^2})$  for D and  $C(q_2) = -\sqrt{u^2 + u_2^2}/(2H)$  for N modes. The off-diagonal matrix  $B$  is nondiagonal with  $B_{[m, q_2]} \equiv B_m(q_2) = \pi H e^{-\sqrt{u^2 + u_2^2}} I_m(ru) [u/(u_2 + \sqrt{u^2 + u_2^2})]^m / \sqrt{u^2 + u_2^2}$  for D and  $B_m(q_2) = (\pi u/H) e^{-\sqrt{u^2 + u_2^2}} I_m'(ru) [u/(u_2 + \sqrt{u^2 + u_2^2})]^{-m}$  for N modes. Here, we have defined the dimensionless combinations  $u = H\sqrt{q_0^2 + q_1^2}$ ,  $u_2 = Hq_2$ , and  $r = R/H$ . The determinant can be obtained straightforwardly, and the total energy can be decomposed to the sum of D and N mode contributions, as

$$E = -\frac{\hbar c L}{H^2} [\Phi^D(r) + \Phi^N(r)], \quad (5)$$

with

$$\Phi^X(r) = -\frac{1}{4\pi} \int_0^\infty du u \ln \{ \det [1 - N^X(u, r)] \}. \quad (6)$$

The matrix  $N^X(u, r)$  is given in terms of Bessel functions,

$$N_{\mu\nu}^D(u, r) = \frac{I_\nu(ru)}{K_\mu(ru)} K_{\mu+\nu}(2u), \quad (7)$$

for D modes and

$$N_{\mu\nu}^N(u, r) = -\frac{I_\nu'(ru)}{K_\mu'(ru)} K_{\mu+\nu}(2u), \quad (8)$$

for  $N$  modes. The determinant in Eq. (6) is taken with respect to the integer indices  $\mu, \nu = -\infty, \dots, \infty$ . If the matrix  $N^X$  is restricted to dimension  $(2l + 1)$  with  $N_{00}^X$  as the central element, it then describes the contribution from  $l$  partial waves, beginning with  $s$  waves for  $l = 0$ .

From Eq. (6), one can easily extract the asymptotic large distance behavior of the energy for  $r = R/H \ll 1$ . For Dirichlet modes  $s$  waves dominate, while for Neumann modes both  $s$  and  $p$  waves ( $l = 1$ ) contribute at leading order in  $r$ . The two cases differ qualitatively, with

$$\Phi^D(r) = -\frac{1}{16\pi} \frac{1}{\ln r}, \quad \text{and} \quad \Phi^N(r) = \frac{5}{32\pi} r^2. \quad (9)$$

For  $H \gg R$  the EM Casimir interaction is dominated by the D (TM) modes. Note that a naïve application of the PFA for small  $r$ , where it is not justified, yields the incorrect scalings  $\Phi^D(r) = \Phi^N(r) \sim r$ .

The natural expectation from the Casimir-Polder result for the plate sphere interaction, that the force at large distance is proportional to the volume of the cylinder, is incorrect. The physical reason for this difference is explained by considering spontaneous charge fluctuations. On a sphere, the positive and negative charges can be separated by at most distances of order  $R \ll H$ . The retarded van der Waals interactions between these dipoles and their images on the plate leads to the Casimir-Polder interaction [6]. In the cylinder, fluctuations of charge along the axis of the cylinder can create arbitrary large positively (or negatively) charged regions. The retarded interaction of these charges (not dipoles) with their images gives the dominant term of the Casimir force. This interpretation is consistent with the difference between the two types of modes, since for N modes such charge modulations cannot occur due to the absence of an electric field along the cylinder axis. Eventually, for a finite cylinder, in the very far region  $H \gg L$ , the charge fluctuations can be considered again as small dipoles, and the Casimir-Polder law is expected to reappear, making the force proportional to the volume of the cylinder  $LR^2$ .

We next consider arbitrary separations, and use Eq. (6) to obtain the contribution from higher order partial waves. A numerical evaluation of the determinant is straightforward, and we find that down to even small separations of  $a/R = 0.1$  the energy converges at order  $l = 25$ , whereas for  $a/R \geq 1$  convergence is achieved for  $l = 4$ . Figure 1 shows our results for Dirichlet and Neumann modes and for their sum which is the EM Casimir energy, all scaled by the corresponding  $E_{\text{PFA}}$  given above [18]. Both types of modes show a strong deviation from the PFA for  $a/R \geq 1$ , especially the Dirichlet energy. Figure 1 shows also very recent word-line-based Monte Carlo results for the Dirichlet case at moderate separations [19], which agree nicely with our exact results.

Note that the asymptotic Dirichlet result of Eq. (9) is also valid for  $R \rightarrow 0$  at fixed  $H$ , i.e., when the cylinder is replaced by an infinitesimal thin wire. The length scale  $R$  is then replaced by a UV cutoff corresponding to some

physical characteristic of the wire. This result illustrates the fact that the asymptotic form of Eq. (9) is *independent* of the actual shape of the cross section of the wire since the cutoff  $R$  can be identified with any typical scale of the cross section. The leading asymptotic term in Eq. (9) is also obtained [8] from the  $s$  wave scattering amplitude for the two-dimensional problem of a strongly repulsive potential concentrated on the wire.

The difference between the D and N modes also appears in the DOS, which in turn affects the temperature dependence of the Casimir force. From Eq. (2) we obtain

$$\delta\rho_X(q_0) = -\frac{q_0HL}{\pi^2} \int_0^\infty \frac{du}{u^2} \times \ln \frac{\det[1 - N^X(\sqrt{u^2 + (q_0H)^2}, r)]}{\det[1 - N^X(q_0H, r)]}, \quad (10)$$

which is convenient both for numerical and analytic computations. Numerical evaluation yields the results shown in Fig. 2 for general values of  $R$ . Analytical results in the limit

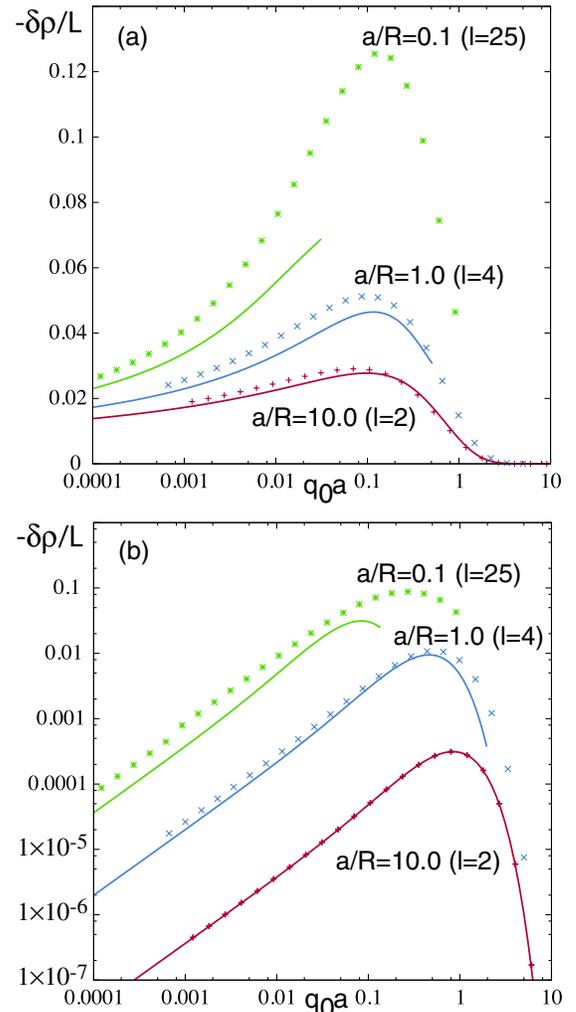


FIG. 2 (color online). The change in the density of states for (a) Dirichlet and (b) Neumann modes obtained from Eq. (10) at order  $l$ . The solid curves show the small  $R$  expansion of Eq. (11).

of small  $R/H$  are obtained by considering the  $s$  waves for D modes, and the  $s$  and  $p$  waves for N modes. For D modes we expand in  $1/\ln(q_0R)$ , whereas for N modes we expand in  $r = R/H$ . To leading order,

$$\delta\rho_D(q_0) = \frac{L}{2\pi} \frac{e^{-2q_0H}}{\ln(q_0R)} + \mathcal{O}[\ln^{-2}(q_0R)], \quad (11a)$$

$$\delta\rho_N(q_0) = -\frac{q_0HL}{8\pi} (1 + 6q_0H)e^{-2q_0H}r^2 + \mathcal{O}(r^3). \quad (11b)$$

Figure 2 allows for an assessment of the validity range of the expansions of Eq. (11) which are shown as solid curves.

These results for the DOS allow us to evaluate for the first time finite temperature contributions to the Casimir interaction in an open geometry. The difference between the free energy  $\mathcal{F}$  and the Casimir energy at  $T = 0$  can be written as [13] ( $k_B$  is Boltzmann constant)

$$\delta\mathcal{F} = \mathcal{F} - E = \pi k_B T \int_0^\infty dq_0 g(q_0) \delta\rho(q_0), \quad (12)$$

with the function  $g(q_0) = \sum_{k=1}^\infty \sin(2\pi k q_0 \lambda_T)/(\pi k)$  and  $\lambda_T = \hbar c/(2\pi k_B T)$ . In the limit  $R \ll (H, \lambda_T)$  but for general  $H/\lambda_T$ , we can use the expansion of Eq. (11) to obtain to leading order in  $1/\ln(R/\lambda_T)$  and  $R/H$  for D and N modes, respectively, the thermal contributions

$$\delta\mathcal{F}_D = \frac{k_B T}{8} \frac{L}{\ln(R/\lambda_T)H} \left[ \coth\left(\frac{H}{\lambda_T}\right) - \frac{\lambda_T}{H} \right], \quad (13a)$$

$$\delta\mathcal{F}_N = -\frac{k_B T}{64} \frac{L\lambda_T R^2}{H^4} \left[ 7 \frac{H}{\lambda_T} \coth\left(\frac{H}{\lambda_T}\right) + \frac{7(H/\lambda_T)^2}{\sinh^2(H/\lambda_T)} + 6 \left(\frac{H}{\lambda_T}\right)^3 \frac{\cosh(H/\lambda_T)}{\sinh^3(H/\lambda_T)} - 20 \right]. \quad (13b)$$

It is interesting to note that  $\delta\mathcal{F}_N$  has a minimum at  $H/\lambda_T = 2.915\dots$ , where the corresponding thermal force changes from repulsive at small  $H$  to attractive at large  $H$ . At low temperatures, the finite  $T$  contributions to the Casimir force  $\delta F = -\partial\delta\mathcal{F}/\partial H$ ,

$$\delta F_D = \frac{2\pi^3}{45} k_B T \left(\frac{k_B T}{\hbar c}\right)^3 \frac{HL}{\ln(R/\lambda_T)}, \quad (14a)$$

$$\delta F_N = \frac{64\pi^5}{945} k_B T \left(\frac{k_B T}{\hbar c}\right)^5 R^2 HL, \quad (14b)$$

have to be added to Eq. (1) for  $H \ll \lambda_T$ . At larger temperatures with  $R \ll \lambda_T \ll H$ , one has the scalings  $\delta F_D \sim k_B T L H^{-2}/\ln(R/\lambda_T)$  and  $\delta F_N \sim -k_B T L R^2 H^{-4}$ . At the extreme high temperature limit of  $\lambda_T \ll R$ , only thermal fluctuations remain, and  $\hbar$  should disappear from the equations. This ‘‘classical limit’’ is well known for parallel plates [20] and is obtained for smooth, arbitrary geometries within the multiple scattering approach [13], and the optical approximation [21]. (Note that for the D modes a subleading  $\hbar$  still survives in the logarithm.)

Finally, we note that our approach can be extended also to multiple wires and distorted beams. Our results should be relevant to nanosystems composed of one-dimensional structures and also to other types of fields as, e.g., thermal order parameter fluctuations.

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