Conservation of Angular Momentum, Transverse Shift, and Spin Hall Effect in Reflection and Refraction of an Electromagnetic Wave Packet

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We present a solution to the problem of reflection and refraction of a polarized Gaussian beam on the interface between two transparent media. The transverse shifts of the beams' centers of gravity are calculated. They always satisfy the total angular momentum conservation law for beams, but, in general, do not satisfy the conservation laws for individual photons as a consequence of the lack of the ''which path'' information in a two-channel wave scattering. The field structure for the reflected and refracted beams is analyzed. In the scattering of a linearly polarized beam, photons of opposite helicities are accumulated at the opposite edges of the beam: this is the spin Hall effect for photons, which can be registered in the cross-polarized component of the scattered beam.

Introduction.—Reflection and refraction of plane electromagnetic waves at the interface between two homogeneous transparent media is described by the Snell law and the Fresnel formulas [1]. However, in the case of the localized wave packets (or beams) the Snell law, as well as the Fresnel formulas (as shown below) give no exact description of their refraction and reflection. First, the reflected packet undergoes a short longitudinal shift in the reflection plane: this is the Goos–Hänchen effect [2], which is not a subject of this work. Besides, a circularly (or elliptically) polarized incident packet experiences a *transverse shift* (TS) and leaves the plane of incidence when refracting or reflecting. This effect was originally predicted by Fedorov [3] and since that time has been discussed in a number of papers, both theoretical [4–7] and experimental [8–10].

TS plays a fundamental role in electrodynamics: this phenomenon is responsible for the conservation of the *total angular momentum* (TAM) of an electromagnetic beam, including the intrinsic (spin) part [5,7]. For a smoothly inhomogeneous medium this effect represents the optical Magnus effect [6,9,11] or the recently discovered topological spin transport (spin Hall effect) of photons [7,12]. However, in spite of a long period of research, up until now the ultimate answer as to the magnitude of TS, along with the correct wording of the TAM conservation law for an electromagnetic beam, has not been found: almost all papers [3–7] result in different answers.

In this Letter we propose an exact solution to the problem of reflection and refraction of an arbitrary polarized Gaussian beam in the framework of classical electrodynamics. This enables us to evaluate TSs of the centers of gravity of the scattered beams, to determine the TAM conservation law that governs the process, and to analyze the field structure in the beams, which reveals the spin Hall effect for photons. It is shown that the mixing of classical

and quantum arguments can lead to an incorrect determination of the beam TS.

Angular momentum conservation laws.—TAM of a polarized electromagnetic wave packet $\mathbf{J}$ consists of the orbital momentum $\mathbf{L}$ and the intrinsic (spin) momentum $\mathbf{S}$. The TAM density (TAM of one photon) can be represented as $\mathbf{j} = \mathbf{r} \times \mathbf{k} + (e|\sigma_3|e)\mathbf{k}/k$ ($\hbar = c = 1$) [7]. Here $\mathbf{r}$, $\mathbf{k}$, and $|e)$ are the radius vector, wave vector, and two-component polarization vector of the wave packet center [$|e)$ is written in the basis of circular polarizations, i.e., helicity basis]; $\sigma_3 = \mathrm{diag}(1, -1)$ is the Pauli matrix. TAM is related to the TAM density as $\mathbf{J} = N\mathbf{j}$, where $N = W/\omega$ is the number of photons in the packet, W is the total field energy of the beam, and ω stands for the frequency (we consider a monochromatic packet). When a wave packet is scattered on the interface $z = 0$ between two homogeneous media, the normal to the surface component of TAM is conserved owing to the axial symmetry of the problem: $J_z^{(i)} = J_z^{(r)} + J_z^{(t)}$. From here on, the superscripts (i), (r), and (t) correspond to the incident, reflected, and refracted wave packets, respectively. The energy conservation law results in the conservation of the total number of photons: $N^{(i)} = N^{(r)} + N^{(t)}$. Taking into account that $W \propto \varepsilon|\mathbf{E}|^2 V$ (ε stands for the permittivity, $\mathbf{E}$ is the electric field in the wave packet, and V is the packet volume), and that the volume of the packet varies as $V \propto n^{-1}|\cos\vartheta|$ in the course of refraction ($n = \sqrt{\varepsilon\mu}$ is the refraction index, while ϑ is the angle between $\mathbf{k}$ and z axis), the conservation law for the z component of TAM reads [5]

$$j_z^{(i)} = R^2 j_z^{(r)} + T^2 \frac{n_2\mu_1\cos\theta'}{n_1\mu_2\cos\theta} j_z^{(t)}. \qquad (1)$$

Here $R, T = |\mathbf{E}^{(r,t)}|/|\mathbf{E}^{(i)}|$ are the Fresnel reflection and refraction coefficients for plane waves, subscripts 1 and 2 refer to parameters of the first and the second medium, μ stands for the permeability, and we denote

$\vartheta^{(i)} = \theta$, $\vartheta^{(t)} = \theta'$, $\vartheta^{(r)} = \pi - \theta$ (Fig. 1). Equation (1) constitutes the main TAM conservation law for a wave packet; it has a classical nature and follows immediately from the Maxwell equations [5].

Equation (1) is inadequate for determining the shifts of the centers of gravity for the reflected or refracted wave packets since one equation contains two unknown terms. Another approach has been suggested in [7], where the authors consider the wave packet scattering as a set of individual reflection and refraction events of isolated photons. In this case each photon finds itself either in the medium 1 (is reflected) or in the medium 2 (is refracted); hence two TAM conservation laws take place for one photon:

$$j_z^{(i)} = j_z^{(r)}, \qquad j_z^{(i)} = j_z^{(t)}. \tag{2}$$

Equation (2) determined the TSs of the wave packets in [7]; numerical simulation for a circularly polarized incident wave packet has shown a good agreement with the theory.

At the same time, Eqs. (2) do *not* always satisfy the main conservation law (1). Equations (1) and (2) coincide only in particular cases: e.g., in the case of total internal reflection, where $R = 1$, $T = 0$, or in the case of small contrast between two media, $|n_2 - n_1| \equiv \delta n \ll 1$. In the latter case $T^2 \frac{n_2 \mu_1 \cos\theta'}{n_1 \mu_2 \cos\theta} = 1 + O(\delta n^2)$, $R^2 = O(\delta n^2)$, and Eq. (1) is equivalent to the second Eq. (2) in the linear approximation in δn, i.e., in the geometrical optics approximation [6,7,11,12]. This clarifies the fact that j_z is an exact integral of motion for the modified equations of geometrical optics in an axially symmetric medium [7]. Below is shown that Eqs. (2) are consistent with Eq. (1) for a circularly polarized initial beam. However, this is not true in the general case of an elliptically polarized beam. We will demonstrate that the scattering of the Gaussian electromagnetic beam in all cases satisfies Eq. (1) and not Eq. (2). The fallacy of Eqs. (2) stems apparently from the quantum-mechanical approach [7], which is based on the events for individual photons. The point is that classical electrodynamics describes a multiphoton interference pattern. By invoking the conservation laws (2) we invoke thereby a "which path" information, which destroys, as is well known from quantum mechanics, the interference pattern in the multiphoton scattering. Thus Eqs. (2) are suitable for describing the scattering process of individual photons; however, in the generic case, they are inapplicable in the scattering of classical wave packets [13].

Gaussian beam reflection and refraction. Transverse shift.—The electric field of the wave packet incident in the plane (x, z) can be represented in the form of a polarized Gaussian beam:

$$\mathbf{E}^{(i)} = A \frac{\mathbf{e}_X + m(\mathbf{e}_y - yB\mathbf{e}_Z)}{\sqrt{1 + |m|^2}} \exp\left(ikZ + \frac{ikBy^2}{2}\right). \tag{3}$$

Here we use a reference frame XyZ associated with the beam (Fig. 1); $\mathbf{e}_{X,y,Z}$ are its unit vectors, the complex value m is related to the beam polarization, the polarization in the
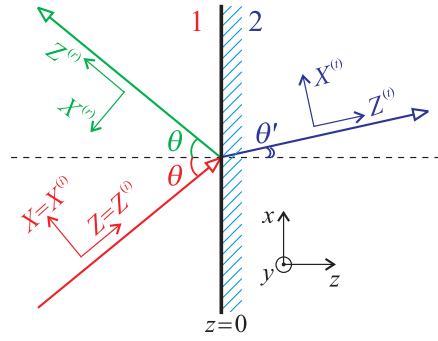

FIG. 1 (color online). The scheme of the wave reflection and refraction with beam coordinates used in the text.

beam center, $y = 0$, is characterized by the polarization vector $(\mathbf{e}_X + m\mathbf{e}_y)/\sqrt{1 + |m|^2}$, or in the helicity basis,

$$|e\rangle = \begin{pmatrix} 1 - im \\ 1 + im \end{pmatrix}\Big/ \sqrt{2(1 + |m|^2)},$$

and the complex parameters $A = A(Z)$ and $B = B(Z)$ vary along the beam in consequence of its diffraction (standard solutions in a homogeneous media are obtainable in the framework of the complex geometrical optics [14]). A is the beam amplitude, while the real and imaginary parts of B are responsible for the phase front curvature and the beam width, respectively.

For the sake of simplicity we assume that the beam (3) is confined in y only, which enables us to consider only TS along this coordinate and not the Goos–Hänchen effect. The deviation from the center, $y \neq 0$, results in a small longitudinal (in the $\mathbf{e}_Z$ direction) field component proportional to y. Owing to this component, field (3) satisfies the Maxwell equations for a homogeneous medium: $\text{div}\mathbf{E} = 0$, i.e., $\mathbf{E}$ is orthogonal to the local wave vector $\mathbf{k}_{\text{loc}}$. It is the longitudinal field component that is responsible for TS of the beam center in the process of beam reflection and refraction. The representation of the Gaussian polarized beam in the form of (3) holds good for sufficiently large distances y until the wavelength is small compared to the characteristic beam width and the radius of curvature of its phase front: $|B|y \ll 1$.

The field of the reflected and refracted beams can be obtained from Eq. (3) supplemented by standard boundary conditions [1]. As a result of cumbersome but direct calculation, the fields for all three beams, (i), (r), and (t), can be written in a unified form:

$$\mathbf{E}^a = \frac{A^a S^a}{\sqrt{1 + |m^a|^2}} \exp\left(ik^a Z^a + \frac{ik^a B^a y^2}{2}\right)$$

$$\times \left\{ \left[1 + \frac{m^a B^a y}{\rho^a \sin\vartheta^a}(\cos\theta - \rho^a \cos\vartheta^a) \right] \mathbf{e}_{X^a} \right.$$

$$\left. + \left[m^a + \frac{B^a y}{\sin\vartheta^a}(\cos\vartheta^a - \rho^a \cos\theta) \right] \mathbf{e}_y - m^a B^a y \mathbf{e}_{Z^a} \right\},$$

$$\tag{4}$$

where $a = (i), (r), (t)$, $S^{(i,r,t)} = 1, R\,\mathrm{sgn}R_\parallel, T$, $\rho^{(i,r,t)} = 1, R_\perp/R_\parallel, T_\perp/T_\parallel$; $m^a = \rho^a m$ is the characteristic of a central polarization of the corresponding beam, $k^{(i)} = k^{(r)} = k$, $k^{(t)} = kn_2/n_1$, $A^{(i)} = A$, $B^{(i)} = B$, $A^{(r,t)}$, and $B^{(r,t)}$ are determined from the boundary conditions $A^{(r,t)}|_{z=0} = A|_{z=0}$ and $B^{(r,t)}|_{z=0} = B|_{z=0}$. For all beams the associated Cartesian coordinates $X^a y Z^a$ and their unit vectors $\mathbf{e}_{X^a,y,Z^a}$ are used (Fig. 1). In the definitions above, we used the Fresnel refraction and reflection coefficients for the plane linearly polarized waves whose electric vector is parallel or orthogonal to the incidence plane [1]: $T_\parallel = 2\varepsilon_1 n_2 \cos\theta/(\varepsilon_2 n_1 \cos\theta + \varepsilon_1 n_2 \cos\theta')$, $T_\perp = 2\mu_2 n_1 \cos\theta/(\mu_2 n_1 \cos\theta + \mu_1 n_2 \cos\theta')$, $R_\parallel = 1 - \cos\theta' T_\parallel/\cos\theta$, $R_\perp = T_\perp - 1$.

To determine the beams' centers of gravity, let us consider the projection of the field (4) onto its central polarization vector [6]: $F = \mathbf{E}^a (\mathbf{e}_{X^a} + m^a \mathbf{e}_y)/\sqrt{1 + |m^a|^2}$. This value can be represented in the first approximation in $|B|y \ll 1$ as $F = A^a S^a \exp[ik^a Z^a + ik^a B^a (y - \delta y^a)^2/2]$, where δy^a is a complex value ($\delta y^{(i)} = 0$). Its imaginary part is responsible for phenomena associated with a phase front curvature (they are omitted in the present study [15]), while $\mathrm{Re}\,\delta y^a = \Delta^a$, represents TS associated with the TAM conservation. Calculations yield

$$\Delta^a = -\frac{\cot\theta}{k} \frac{\mathrm{Im}\,m(1 + \rho^{a2} - 2\rho^a \cos\vartheta^a/\cos\theta)}{1 + \rho^{a2}|m|^2}. \quad (5)$$

Formulas for $T_{\parallel,\perp}$ and $R_{\parallel,\perp}$ have been considered real, which excludes the case of total internal reflection, where $\cos\theta' = \sqrt{1 - (n_1^2/n_2^2)\sin^2\theta}$ becomes imaginary. Calculations for the totally reflected beam read

$$\Delta^{(\mathrm{tot}\,r)} = -\frac{2\cot\theta}{k} \frac{\mathrm{Im}\,m(1 + \mathrm{Re}\,\rho^{(r)}) + \mathrm{Re}\,m\,\mathrm{Im}\,\rho^{(r)}}{1 + |m|^2}. \quad (6)$$

Let us verify now the agreement between Eqs. (5) and (6) and the TAM conservation laws (1) and (2). The z component of the beam's TAM density equals [5,7] $j_z^a = -\Delta^a k^a \sin\vartheta^a + \frac{2\,\mathrm{Im}\,m^a}{1 + |m^a|^2}\cos\vartheta^a$. By substituting these values along with TSs (5) and (6), and $R = \sqrt{|R_\parallel|^2 + |R_\perp|^2 |m|^2}/\sqrt{1 + |m|^2}$, $T = \sqrt{|T_\parallel|^2 + |T_\perp|^2 |m|^2}/\sqrt{1 + |m|^2}$ (or $T = 0$ for the total internal reflection) into (1), we make sure that the TAM conservation law (1) is satisfied identically. At the same time, shifts (5) and (6) satisfy the TAM conservation laws for individual photons, Eqs. (2), solely in the following particular situations: (A) the incident beam is linearly polarized, $\mathrm{Im}\,m = 0$, and $\Delta^a = 0$; (B) the incident beam is circularly polarized, $m = \pm i$ [this explains a good agreement of numerical simulation in [7] with Eqs. (2)]; (C) the case of total internal reflection, where just one scattering channel exists and laws (1) and (2) are identical. In all other cases shifts (5) do not satisfy Eqs. (2).

Spin Hall effect of light.—Equation (4) enables one not only to find TS of the centers of gravity but also to determine the field structure in the reflected and refracted beams. One can see that the reflected and refracted beams do not have the form of the Gaussian beams (3) shifted in accordance with Eqs. (5) and (6). However, in the first approximation in $|B|y \ll 1$, field (4) can be represented as a superposition of two Gaussian circularly polarized beams like (3): $\mathbf{E}^a = \alpha^+ \mathbf{E}^{a+} + \alpha^- \mathbf{E}^{a-}$, where $\alpha^\pm = (1 \mp im^a)/\sqrt{2(1 + |m^a|^2)}$ and $\mathbf{E}^{a\pm} = A^a S^a \frac{\mathbf{e}_{X^a} \pm i(\mathbf{e}_y - yB^a \mathbf{e}_{Z^a})}{\sqrt{2}} \times \exp[ik^a Z^a + \frac{ik^a B^a (y - \delta^{a\pm})^2}{2}]$. Assume that the incident beam is linearly polarized with the electric field parallel or orthogonal to the incidence plane: $m_{\parallel,\perp} = 0, \infty$. In this case $\Delta^a = 0$; however, the shifts of the reflected and refracted partial beams $\mathbf{E}^{a\pm}$ are nonzero and oppositely directed: $\delta_\parallel^{a\pm} = \pm(\cos\vartheta^a - \rho^a \cos\theta)/k \sin\theta$ and $\delta_\perp^{a\pm} = \pm(\cos\vartheta^a - \rho^{a^{-1}}\cos\theta)/k \sin\theta$ (here we do not consider the total internal reflection case). This confirms the predicted earlier effect of splitting of a beam of mixed polarization into two circularly polarized beams in an inhomogeneous medium [11,12]. The splitting is very small—fractions of the wavelength; nevertheless, it leads to new observable phenomena.

Indeed, the elliptical polarizations of opposite signs arise at the opposite edges of the beam. (As a consequence, the beam as a whole is depolarized, i.e., in contrast to the Fresnel formulas for plane waves, the polarization state of the linearly polarized beam changes after reflection and refraction and becomes mixed.) In the approximation considered, the degree of the circular polarization is proportional to y, i.e., it grows linearly with the distance from the beam center. The initiation of elliptical polarizations at the ends of a linearly polarized beam is a manifestation of the spin Hall effect for photons: the photons of opposite helicities accumulate at the opposite ends of the beam just as in the recently discovered spin Hall effect for carriers in semiconductors [16,17]. It is interesting that this effect for photons was predicted as early as 1965 in the paper of Costa de Beauregard [4].

The change in the polarization structure along with the splitting of a linearly polarized beam can be observed experimentally by measuring the cross-polarized field component of the reflected and refracted beam (i.e., E_y^a and $E_{X^a}^a$ for $m = 0$ and ∞, respectively). The intensity of the cross component, being closely related to the degree of the circular polarization, is equal to $I_{\mathrm{cross}}^a = I^a |B^a k^a \delta^{a+}|^2 y^2 \propto y^2 \exp(-\mathrm{Im}\,B^a y^2)$, where $I^a = |A^a|^2 S^{a2} \exp(-\mathrm{Im}\,B^a y^2)$ is the field intensity in the beam. The relative cross-component intensity grows infinitely with y: $I_{\mathrm{cross}}^a/I^a \propto y^2$. Figure 2 presents the distributions of absolute and relative cross-component intensities in the reflected beam. The beam splitting is easily visible, while the angle corresponding to the maximum of the absolute cross-component intensity visually coincides with the angle associated with the maximum TS of the circularly polarized beam [7]. A flip of the helicity [Fig. 2(a)] and a singularity in the relative cross-component intensity in the relative cross-component intensity [Fig. 2(e)] for $m = 0$
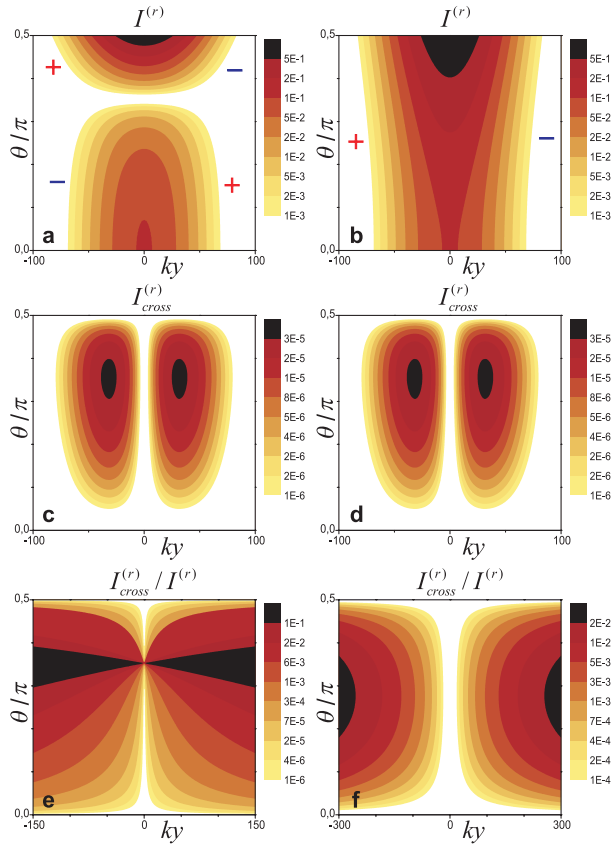
FIG. 2 (color online). Intensity of the reflected beam's field (a),(b), and absolute (c),(d) as well as relative (e),(f) intensity of the cross-polarized component in the beam via the transverse coordinate and incidence angle at different polarizations of the initial beam [$m = 0$ for (a,c,e) and $m = \infty$ for (b,d,f)]. The signs of the elliptical polarizations at the different sides of the beam are marked (a),(b). Parameters are: $n_2/n_1 = 2$, $\mu_{1,2} = 1$, $A^{(r)} = 1$, and $k^{-1}B^{(r)} = i10^{-3}$ in the observation point, which corresponds to the beam's width about 10 wavelengths.

occur at the Brewster angle where in-plane component $E_{X^{(r)}}^{(r)}$ vanishes and changes its sign. With the characteristic parameters of present-day optical polarizers and laser beams, one can look forward to detect the cross component like that in Fig. 2 and, hence, to register the spin Hall effect of photons.

Conclusion.—We have solved the problem of reflection and refraction of an electromagnetic polarized Gaussian beam at the interface between two homogeneous media. The transverse shifts of the centers of gravity for the reflected and refracted beams have been calculated. In all cases they satisfy the total angular momentum conservation law for beams, but in the generic case do not satisfy the conservation laws for individual photons because of the fundamentally two-channel character of the process and the lack of "which path" information in classical electrodynamics. The field structure for the reflected and refracted beams has been analyzed. An initially linearly polarized beam splits into two circularly polarized beams shifted in

opposite directions. This causes the rise of elliptical polarizations of opposite signs at the beam edges, i.e., the spin Hall effect for photons. The effect can be detected by measuring the split cross component of the scattered beam's field.

[1] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975).
[2] F. Goos and H. Hänchen, Ann. Phys. (Berlin) **1**, 333 (1947).
[3] F. I. Fedorov, Dokl. Akad. Nauk SSSR **105**, 465 (1955).
[4] O. Costa de Beauregard, Phys. Rev. **139**, B1443 (1965); H. Schilling, Ann. Phys. (Berlin) **16**, 122 (1965); J. Ricard, Nouv. Rev. Opt. Appl. **1**, 273 (1970); D. G. Boulware, Phys. Rev. D **7**, 2375 (1973); N. Ashby and S. C. Miller, Jr., Phys. Rev. D **7**, 2383 (1973).
[5] M. A. Player, J. Phys. A **20**, 3667 (1987); V. G. Fedoseev, *ibid.* **21**, 2045 (1988); W. N. Hugrass, J. Mod. Opt. **37**, 339 (1990); V. G. Fedoseev, Opt. Spectrosc. (USSR) **71**, 483 (1991); Opt. Spectrosc. (USSR) **71**, 570 (1991).
[6] V. S. Liberman and B. Ya. Zel'dovich, Phys. Rev. A **46**, 5199 (1992).
[7] M. Onoda, S. Murakami, and N. Nagaosa, Phys. Rev. Lett. **93**, 083901 (2004).
[8] C. Imbert, Phys. Rev. D **5**, 787 (1972).
[9] A. V. Dooghin *et al.*, Phys. Rev. A **45**, 8204 (1992).
[10] F. Pillon, H. Gilles, and S. Girard, Appl. Opt. **43**, 1863 (2004).
[11] K. Yu. Bliokh and Yu. P. Bliokh, Phys. Rev. E **70**, 026605 (2004); JETP Lett. **79**, 519 (2004).
[12] K. Yu. Bliokh and Yu. P. Bliokh, Phys. Lett. A **333**, 181 (2004); K. Yu. Bliokh and V. D. Freilikher, Phys. Rev. B **72**, 035108 (2005); A. Bérard and H. Mohrbach, hep-th/0404165.
[13] When a single photon is scattered, it can be either reflected or refracted, the observer evidently has "which path" information, and Eq. (2) holds true. If a number of photons experience scattering simultaneously, the observer has no chance to establish a one-to-one correspondence between incident and reflected or refracted photons, the interference takes place and only the total angular momentum is conserved, Eq. (1).
[14] Yu. A. Kravtsov, G. W. Forbes, and A. A. Asatryan, Prog. Opt. **39**, 1 (1999).
[15] A shift of the beam's center of gravity, which is related to $\text{Im}\delta y^a \propto \text{Re}m$, can be excluded either by considering the beam in the point of flat phase front or (not in the total internal reflection case) by assuming $\text{Re}m = 0$. The latter means that the polarization ellipse of the incident beam is oriented along x and y coordinate axes.
[16] S. Murakami, N. Nagaosa, and S.-C. Zhang, Science **301**, 1348 (2003); J. Sinova *et al.*, Phys. Rev. Lett. **92**, 126603 (2004); see also K. Yu. Bliokh and Yu. P. Bliokh, Ann. Phys. (N.Y.) **319**, 13 (2005), and references therein.
[17] Y. K. Kato *et al.*, Science **306**, 1910 (2004); (see also B. A. Bernevig and S.-C. Zhang, cond-mat/0412550); J. Wunderlich *et al.*, Phys. Rev. Lett. **94**, 047204 (2005).