Teleportation and Dense Coding with Genuine Multipartite Entanglement

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We present an explicit protocol $\mathcal{E}_0$ for faithfully teleporting an arbitrary two-qubit state via a genuine four-qubit entangled state. By construction, our four-partite state is not reducible to a pair of Bell states. Its properties are compared and contrasted with those of the four-party Greenberger-Horne-Zeilinger and W states. We also give a dense coding scheme $\mathcal{D}_0$ involving our state as a shared resource of entanglement. Both $\mathcal{D}_0$ and $\mathcal{E}_0$ indicate that our four-qubit state is a likely candidate for the genuine four-partite analogue to a Bell state.

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Quantum teleportation, the disembodied transport of quantum states between subsystems through a classical communication channel requiring a shared resource of entanglement, is one of the most profound results of quantum information theory [1]. Bennett *et al.* [2] are the first to show how quantum entanglement can assist in the teleportation of an intact quantum state

$$|\psi\rangle_{A_1} = a|0\rangle_{A_1} + b|1\rangle_{A_1}, \tag{1}$$

with $a, b \in C^1$ and $|a|^2 + |b|^2 = 1$, from one place to another, by a sender, Alice, who knows neither the state $|\psi\rangle_{A_1}$ to be teleported nor the location of the intended receiver, Bob. In their standard teleportation protocol $\mathcal{T}_0$, Alice and Bob share *a priori* a pair of particles, A_2 and B, in a maximally entangled Bell state, say

$$|\Psi^0_{\rm Bell}\rangle_{A_2B} \equiv \frac{1}{\sqrt{2}}(|00\rangle_{A_2B} + |11\rangle_{A_2B}). \tag{2}$$

Teleportation firmly establishes the practical basis for considering the maximally entangled Bell states as basic units, upon which bipartite entanglement can be quantitatively expressed in terms of. Indeed, quantities like the concurrence [3,4] and fully entangled fraction [5] have their roots in these states.

The teleportation of an arbitrary two-qubit state,

$$|\Psi\rangle_{A_1A_2} = \sum_{i,j=0}^{1} a_{ij}|ij\rangle_{A_1A_2}, \tag{3}$$

with $a_{i,j} \in C^1$ and $\sum_{i,j=0}^1 |a_{ij}|^2 = 1$, had been studied by Lee *et al.* [6] and recently by Rigolin [7]. Whereas Lee *et al.* did not explicitly construct a protocol, the 16G states defined by Rigolin in his protocol $|G^{ij}\rangle_{A_3A_4B_1B_2} \equiv [(\sigma^i_{A_3} \otimes \sigma^j_{A_4}) \otimes I_{B_1B_2}]|G^{00}\rangle_{A_3A_4B_1B_2}$, with $|G^{00}\rangle \equiv (|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle)/2$, were actually tensor products of two Bell states:

$$|G^{00}\rangle_{A_3A_4B_1B_2} = |\Psi^0_{\rm Bell}\rangle_{A_3B_1} \otimes |\Psi^0_{\rm Bell}\rangle_{A_4B_2}. \tag{4}$$

This fact had been highlighted in Ref. [6]. In this Letter, we give an explicit protocol $\mathcal{E}_0$ for faithfully teleporting arbitrary two-qubit states employing genuine four-qubit entangled states $|\bar{\chi}^{00}\rangle$ [Eq. (9)], and especially $|\chi^{00}\rangle$ [Eq. (18)]. This is an important consideration because $|\chi^{00}\rangle$, in addition to $|\Psi^0_{\rm Bell}\rangle \otimes |\Psi^0_{\rm Bell}\rangle$, could be a likely candidate for the genuine four-partite analogue to $|\Psi^0_{\rm Bell}\rangle$. Since our work is motivated in part by $\mathcal{T}_0$, we briefly describe it below before presenting our protocol $\mathcal{E}_0$. This is followed by a detailed analysis on the entanglement properties of $|\bar{\chi}^{00}\rangle$ and $|\chi^{00}\rangle$, where we compare and contrast with those of the four-party Greenberger-Horne-Zeilinger (GHZ) [8] and W [9] states. Before concluding, we give a dense coding scheme $\mathcal{D}_0$ using $|\chi^{00}\rangle$ as the shared resource of entanglement.

In $\mathcal{T}_0$, the initial complete state of the three particles, A_1, A_2, and B, is a pure product state,

$$|\psi\rangle_{A_1}\langle\psi| \otimes |\Psi^0_{\rm Bell}\rangle_{A_2B}\langle\Psi^0_{\rm Bell}|, \tag{5}$$

involving neither classical correlation nor quantum entanglement between particle A_1 and the maximally entangled pair A_2B. Alice cleanly divides the full information encoded in $|\psi\rangle_{A_1}$ into two parts, transmitting first the purely nonclassical part via the quantum channel $|\Psi^0_{\rm Bell}\rangle_{A_2B}$, by performing a complete von Neumann measurement in the Bell basis:

$$|\Psi^i_{\rm Bell}\rangle_{A_1A_2} = (\sigma^i_{A_1} \otimes \sigma^0_{A_2})|\Psi^0_{\rm Bell}\rangle_{A_1A_2}, \tag{6}$$

on the joint system consisting of particles A_1 and A_2. Here, $\sigma^0 = I_2$ is the two-dimensional identity and $\sigma^i (i = 1, 2, 3)$ are the Pauli matrices. We emphasize that it is a consequence of the fact that $|\Psi^0_{\rm Bell}\rangle_{A_1A_2}$ is maximally entangled, that the $|\Psi^i_{\rm Bell}\rangle_{A_1A_2}$'s are obtainable from $|\Psi^0_{\rm Bell}\rangle_{A_1A_2}$ by appropriate local one-particle Pauli rotation. The density operator of Bob's qubit ρ^i_B conditioned on Alice's Bell measurement outcome i is

$$\frac{1}{p_i}\,\mathrm{tr}_{A_1A_2}[(|\psi\rangle_{A_1}\langle\psi|\otimes|\Psi^0_{\mathrm{Bell}}\rangle_{A_2B}\langle\Psi^0_{\mathrm{Bell}}|)(|\Psi^i_{\mathrm{Bell}}\rangle_{A_1A_2}\langle\Psi^i_{\mathrm{Bell}}|\otimes I_B)]$$
$$=\frac{1}{p_i}\,{}_{A_1A_2}\langle\Psi^0_{\mathrm{Bell}}|(\sigma^i_{A_1}|\psi\rangle_{A_1}\otimes|\Psi^0_{\mathrm{Bell}}\rangle_{A_2B})({}_{A_1}\langle\psi|\sigma^i_{A_1}\otimes{}_{A_2B}\langle\Psi^0_{\mathrm{Bell}}|)|\Psi^0_{\mathrm{Bell}}\rangle_{A_1A_2}=\frac{1}{4p_i}\sigma^i_B|\psi\rangle_B\langle\psi|\sigma^i_B, \tag{7}$$

where $p_i = \mathrm{tr}[(|\psi\rangle_{A_1}\langle\psi|\otimes|\Psi^0_{\mathrm{Bell}}\rangle_{A_2B}\langle\Psi^0_{\mathrm{Bell}}|)(|\Psi^i_{\mathrm{Bell}}\rangle_{A_1A_2}\times\langle\Psi^i_{\mathrm{Bell}}|\otimes I_B)] = 1/4$. It follows that, regardless of the unknown state $|\psi\rangle_{A_1}$, the four measurement outcomes are equally likely. Alice gains no information about the state $|\psi\rangle_{A_1}$ from her measurement. She is left with particles A_1 and A_2 in some maximally entangled Bell state, without any trace of the original $|\psi\rangle_{A_1}$. The outcome of Alice's measurement constitutes the second purely classical part of the full information encoded in $|\psi\rangle_{A_1}$. She communicates this 2 bits of information via a classical channel, after which Bob applies the required Pauli rotation to transform the state of his particle B into an accurate replica of the original state of Alice's particle A_1. Equation (7) follows from, and the success of $\mathcal{T}_0$ is guaranteed by, the following identity. For the maximally entangled state Eq. (2), we have [10]

$$\begin{aligned}{}_{A_1A_2}\langle\Psi^0_{\mathrm{Bell}}|\Psi^0_{\mathrm{Bell}}\rangle_{A_2B} &= \frac{1}{2}\sum_{i,j=0}^{1}({}_{A_1}\langle i|\otimes{}_{A_2}\langle i|)(|j\rangle_{A_2}\otimes|j\rangle_B)\\ &=\frac{1}{2}\sum_{i=0}^{1}|i\rangle_B\times{}_{A_1}\langle i|.\end{aligned} \tag{8}$$

Our protocol $\mathcal{E}_0$ is motivated, in particular, by Eqs. (4) and (8). To avoid our four-qubit entangled channel from being reducible to a tensor product of two Bell states, and to ensure the success of faithfully teleporting any arbitrary two-qubit state, Alice and Bob share *a priori* two pairs of particles, A_3A_4 and B_1B_2, in the state

$$|\bar{\chi}^{00}\rangle_{A_3A_4B_1B_2}\equiv\frac{1}{2}\sum_{J=0}^{3}|J\rangle_{A_3A_4}\otimes|J'\rangle_{B_1B_2}. \tag{9}$$

The $|J\rangle$'s constitute an orthonormal basis, and explicitly

$$\begin{aligned}|0\rangle &= \cos\theta_1|00\rangle+\sin\theta_1|11\rangle,\\ |1\rangle &= \cos\phi_1|01\rangle+\sin\phi_1|10\rangle,\\ |2\rangle &= -\sin\phi_1|01\rangle+\cos\phi_1|10\rangle,\\ |3\rangle &= -\sin\theta_1|00\rangle+\cos\theta_1|11\rangle.\end{aligned} \tag{10}$$

The $|J'\rangle$'s constitute another orthonormal basis:

$$\begin{aligned}|0'\rangle &= \cos\theta_2|00\rangle+\sin\theta_2|11\rangle,\\ |1'\rangle &= \sin\phi_2|01\rangle+\cos\phi_2|10\rangle,\\ |2'\rangle &= \cos\phi_2|01\rangle-\sin\phi_2|10\rangle,\\ |3'\rangle &= -\sin\theta_2|00\rangle+\cos\theta_2|11\rangle.\end{aligned} \tag{11}$$

Here, $0<\theta_1, \theta_2, \phi_1, \phi_2<\pi/2$, and we demand that $\theta_1\neq\theta_2$, $\phi_1\neq\phi_2$. In particular, we may express

$$|\Psi\rangle_{A_1A_2}=\sum_{J=0}^{3}\alpha_J|J'\rangle_{A_1A_2}, \tag{12}$$

with $\alpha_J\in C^1$ and $\sum_{J=0}^{3}|\alpha_J|^2=1$. By virtue of the fact that, between A_3A_4 and B_1B_2, $|\bar{\chi}^{00}\rangle_{A_3A_4B_1B_2}$ is a maximally entangled state [compare with Eq. (2)], we may construct the following basis of 16 orthonormal states [similar to Eq. (6)]:

$$\begin{aligned}|\bar{\Pi}^{00}\rangle_{A_1A_2A_3A_4} &\equiv\frac{1}{2}\sum_{K=0}^{3}|K'\rangle_{A_1A_2}\otimes|K\rangle_{A_3A_4},\\ |\bar{\Pi}^{ij}\rangle_{A_1A_2A_3A_4} &= [(\sigma^i_{A_1}\otimes\sigma^j_{A_2})\otimes I_{A_3A_4}]|\bar{\Pi}^{00}\rangle_{A_1A_2A_3A_4}.\end{aligned} \tag{13}$$

If Alice performs a complete projective measurement jointly on $A_1A_2A_3A_4$ in the above basis with the measurement outcome ij, then Bob's pair of particles B_1B_2 will be in the state

$$\begin{aligned}&\frac{1}{\sqrt{p_{ij}}}\,{}_{A_1A_2A_3A_4}\langle\bar{\Pi}^{ij}|(|\Psi\rangle_{A_1A_2}\otimes|\bar{\chi}^{00}\rangle_{A_3A_4B_1B_2})\\ &\quad=\frac{1}{\sqrt{p_{ij}}}\,{}_{A_1A_2A_3A_4}\langle\bar{\Pi}^{00}|[(\sigma^i_{A_1}\otimes\sigma^j_{A_2})|\Psi\rangle_{A_1A_2}\\ &\qquad\otimes|\bar{\chi}^{00}\rangle_{A_3A_4B_1B_2}]\\ &\quad=\frac{1}{4\sqrt{p_{ij}}}(\sigma^i_{B_1}\otimes\sigma^j_{B_2})|\Psi\rangle_{B_1B_2}.\end{aligned} \tag{14}$$

Here, $|\Psi\rangle_{A_1A_2}\otimes|\bar{\chi}^{00}\rangle_{A_3A_4B_1B_2}$ is the initial complete state of the six particles, A_1, A_2, A_3, A_4, B_1, and B_2. Equation (14) is the analogue of Eq. (7). And, as in $\mathcal{T}_0$, the success of $\mathcal{E}_0$ is guaranteed by the following identity:

$${}_{A_1A_2A_3A_4}\langle\bar{\Pi}^{00}|\bar{\chi}^{00}\rangle_{A_3A_4B_1B_2}=\frac{1}{4}\sum_{J,K=0}^{3}({}_{A_1A_2}\langle K'|\otimes{}_{A_3A_4}\langle K|)(|J\rangle_{A_3A_4}\otimes|J'\rangle_{B_1B_2})=\frac{1}{4}\sum_{J=0}^{3}|J'\rangle_{B_1B_2}\times{}_{A_1A_2}\langle J'|. \tag{15}$$

Clearly, $p_{ij}=1/16$ and Bob will always succeed in recovering an exact replica of the original state Eq. (12) of Alice's particles A_1A_2, upon receiving 4 bits of classical information about her measurement result.

Now, let us consider the entanglement properties of

$$|\bar{\chi}^{00}\rangle_{A_3A_4B_1B_2}=\frac{1}{\sqrt{2}}(|\bar{\zeta}^0\rangle+|\bar{\zeta}^1\rangle)_{A_3A_4B_1B_2}, \tag{16}$$

with $|\bar{\zeta}^0\rangle \equiv (\cos\theta_{12}|0000\rangle - \sin\theta_{12}|0011\rangle - \sin\phi_{12}|0101\rangle + \cos\phi_{12}|0110\rangle)/\sqrt{2}$ and $|\bar{\zeta}^1\rangle \equiv (\cos\phi_{12}|1001\rangle + \sin\phi_{12}|1010\rangle + \sin\theta_{12}|1100\rangle + \cos\theta_{12}|1111\rangle)/\sqrt{2}$. By inspection, we would also have maximum entanglement between A_3B_1 and A_4B_2 if we demand that $\phi_{12} \equiv \phi_1 - \phi_2 = \theta_1 - \theta_2 \equiv \theta_{12}$. In contrast, for a pair of Bell states, there is zero entanglement between A_3B_1 and A_4B_2. Thus, in this sense, the resulting state is "maximally" different from a pair of Bell states. Furthermore, the amount of entanglement between A_3B_2 and A_4B_1 is given by the von Neumann entropy

$$S[\rho_{A_3B_2}] = -\cos^2\theta_{12}\log_2\cos^2\theta_{12} - \sin^2\theta_{12}\log_2\sin^2\theta_{12}, \tag{17}$$

where $\rho_{A_3B_2} = \mathrm{tr}_{A_4B_1}(|\bar{\chi}^{00}\rangle_{A_3A_4B_1B_2}\langle\bar{\chi}^{00}|)$. Clearly, $S[\rho_{A_3B_2}]$ has maximum value 1 when $\theta_{12} = \pi/4$. Imposing these conditions, we obtain

$$|\chi^{00}\rangle_{A_3A_4B_1B_2} = \frac{1}{\sqrt{2}}(|\zeta^0\rangle + |\zeta^1\rangle)_{A_3A_4B_1B_2}, \tag{18}$$

with $|\zeta^0\rangle \equiv (|0000\rangle - |0011\rangle - |0101\rangle + |0110\rangle)/2$ and $|\zeta^1\rangle \equiv (|1001\rangle + |1010\rangle + |1100\rangle + |1111\rangle)/2$. From Eq. (18), we can generate a basis of 16 orthonormal states either by applying σ^i and σ^j to A_3 and A_4, respectively [as in Eq. (13)], or to A_3 and B_1, respectively, since A_3B_1 and A_4B_2 are maximally entangled too. However, we cannot generate the desired basis by applying σ^i and σ^j to A_3 and B_2, respectively, since A_3B_2 and A_4B_1 are not maximally entangled. Instead, we may have, for instance, the following orthonormal basis:

$$\{(\sigma^0_{A_3} \otimes \sigma^j_{B_2})|\chi^{00}\rangle_{A_3A_4B_1B_2}, \quad (\sigma^3_{A_3} \otimes \sigma^j_{B_2})|\chi^{00}\rangle_{A_3A_4B_1B_2}\} \tag{19}$$

for an eight-dimensional subspace. If we consider $\mathcal{E}_0$ for an arbitrary two-qubit state via A_3B_2 to A_4B_1, the state of particles A_4B_1 conditioned on Alice's measurement result ij:

$$\begin{aligned}&\frac{1}{\sqrt{p_{ij}}}{}_{A_1A_2A_3B_2}\langle\Pi^{ij}|(|\Psi\rangle_{A_1A_2} \otimes |\chi^{00}\rangle_{A_3A_4B_1B_2})\\ &\quad = \frac{1}{\sqrt{p_{ij}}}{}_{A_1A_2A_3B_2}\langle\Pi^{00}|[(\sigma^i_{A_1} \otimes \sigma^j_{A_2})|\Psi\rangle_{A_1A_2}\\ &\qquad \otimes |\chi^{00}\rangle_{A_3A_4B_1B_2}],\end{aligned} \tag{20}$$

where it follows from Eq. (13):

$$|\Pi^{00}\rangle_{A_1A_2A_3B_2} = \frac{1}{\sqrt{2}}(|\Phi^0\rangle + |\Phi^1\rangle)_{A_1A_2A_3B_2}, \tag{21}$$

with $|\Phi^0\rangle \equiv (|0000\rangle + |0011\rangle - |0101\rangle + |0110\rangle)/2$ and $|\Phi^1\rangle \equiv (|1001\rangle + |1010\rangle - |1100\rangle + |1111\rangle)/2$, which together with Eq. (18) yield [in contrast to Eq. (15)],

$$\begin{aligned}{}_{A_1A_2A_3B_2}\langle\Pi^{00}|\chi^{00}\rangle_{A_3A_4B_1B_2} &= \frac{1}{4}[|\Psi^0_{\rm Bell}\rangle_{A_4B_1} \times {}_{A_1A_2}(\langle\Psi^0_{\rm Bell}|\\ &\quad + \langle\Psi^3_{\rm Bell}|) + |\Psi^1_{\rm Bell}\rangle_{A_4B_1}\\ &\quad \times {}_{A_1A_2}(\langle\Psi^2_{\rm Bell}| + \langle\Psi^2_{\rm Bell}|].\end{aligned} \tag{22}$$

This implies that faithful teleportation is possible only for partially unknown entangled states such as $\alpha_0|\Psi^0_{\rm Bell}\rangle_{A_1A_2} + \alpha_1|\Psi^1_{\rm Bell}\rangle_{A_1A_2}$. From here on, we focus our analysis on $|\chi^{00}\rangle_{A_3A_4B_1B_2}$.

By construction, there is absolutely zero entanglement between any one particle and any other particle. The entanglement is purely between pairs of particles: A_3A_4 and B_1B_2, A_3B_1 and A_4B_2, and A_3B_2 and A_4B_1. This is in contrast to two Bell pairs where the maximal entanglement between A_3A_4 and B_1B_2 is due to those between $A_3(A_4)$ and $B_1(B_2)$. The behavior of the entanglement associated with $|\chi^{00}\rangle_{A_3A_4B_1B_2}$ under particle loss resembles that of a GHZ state [8,11], in that

$$S[\sigma] = 1, \tag{23}$$

where σ is the resultant density operator from partial tracing $|\chi^{00}\rangle_{A_3A_4B_1B_2}$ over any one of the four particles; i.e., the lost particle is in a completely mixed state. Incidentally, one can teleport perfectly an arbitrary qubit from any one party to any other party if the other two parties choose to cooperate as in the teleportation protocol of Karlsson *et al.* [12], which employs a GHZ channel:

$$|\psi\rangle_{A_1} \otimes |\chi^{00}\rangle_{A_2B_1B_2B_3} = \frac{1}{2}\sum_{i=0}^{3}|\Psi^i_{\rm Bell}\rangle_{A_1A_2} \otimes |\Theta^i\rangle_{B_1B_2B_3}, \tag{24}$$

where $|\Theta^{0,3}\rangle \equiv a|\Xi^0\rangle \pm b|\Xi^1\rangle$, $|\Theta^{1,2}\rangle \equiv a|\Xi^1\rangle \pm b|\Xi^0\rangle$, with $|\Xi^0\rangle \equiv (|000\rangle - |011\rangle - |101\rangle + |110\rangle)/2$ and $|\Xi^1\rangle \equiv (|001\rangle + |010\rangle + |100\rangle + |111\rangle)/2$. In particular, if B_1 and B_2 measure in the $\{|0\rangle, |1\rangle\}$ basis, and together with Alice communicate classically their measurement results to B_3, he would be able to obtain $|\psi\rangle_{B_3}$. It is not difficult to see that the protocol works because measurements in the $\{|0\rangle, |1\rangle\}$ basis carried out by any two parties on $|\chi^{00}\rangle$ establish a Bell channel across the other two parties.

We note that σ is entangled, whereas any reduced state obtained from a GHZ state is separable. Specifically, if particle A_3 is lost, the nonzero negativity [13] between A_4 and B_1B_2 is equal to that between B_1 and A_4B_2. This is surprising because the original entanglement was between the pairs of particles, yet it is not completely destroyed due to particle loss. In this sense, the behavior of the entanglement associated with $|\chi^{00}\rangle_{A_3A_4B_1B_2}$ under particle loss also resembles that of a W state [11,9]. However, a further particle loss will destroy all entanglement.

Lastly, $|\chi^{00}\rangle$ truly differs from the four-qubit GHZ and W states in that both these states do not enable the teleportation of an arbitrary two-qubit state. Indeed, they are inequivalent under stochastic local operations and classical communication. The sixth-order four-qubit filter $\mathcal{F}_3^{(4)}$ [14] has nonzero expectation value for $|\chi^{00}\rangle$:

$$\langle\chi^{00}|\mathcal{F}_3^{(4)}|\chi^{00}\rangle \equiv \frac{1}{2}\sum_{\alpha,\beta,\gamma=0}^{3} E^{\alpha_1\alpha_2}E_{\alpha_1\alpha_2}E^{\beta_1\beta_2}E_{\beta_1\beta_2}E^{\gamma_1\gamma_2}E_{\gamma_1\gamma_2} = 1.$$

Here, $E^{\alpha_1\alpha_2} \equiv \langle\chi^{00}|\sigma^{\alpha_1}\otimes\sigma^{\alpha_2}\otimes\sigma^2|\chi^{00}\rangle$, $E^{\beta_1\beta_2} \equiv \langle\chi^{00}|\sigma^{\beta_1}\otimes\sigma^2\otimes\sigma^{\beta_2}\otimes\sigma^2|\chi^{00}\rangle$, $E^{\gamma_1\gamma_2} \equiv \langle\chi^{00}|\sigma^2\otimes\sigma^{\gamma_1}\otimes\sigma^{\gamma_2}\otimes\sigma^2|\chi^{00}\rangle$, and $E_{\kappa\lambda} = g_{\kappa\mu}g_{\lambda\nu}E^{\mu\nu}$ with $g_{\mu\nu} \equiv \mathrm{diag}\{-1, 1, 0, 1\}$. It has the value 1/2 for the GHZ state and 0 for the W state. On the other hand, the third order filter $\mathcal{F}_1^{(4)}$ and fourth order filter $\mathcal{F}_2^{(4)}$ have expectation value 1 for the GHZ state but yield, respectively, 0 and 1 for $|\chi^{00}\rangle$. Therefore, $|\chi^{00}\rangle$ is a "new" genuine multipartite entangled state. It is not distinguished by the classification for pure four-qubit states of Ref. [15]. Note that we are not claiming that $|\chi^{00}\rangle$ is LOCC inequivalent to either the GHZ or W state. This would require further work. For now, we turn our attention to dense coding [16].

A dense coding scheme $\mathcal{D}_0$ using $|\chi^{00}\rangle_{A_3A_4B_1B_2}$, which "mirrors" $\mathcal{E}_0$ is the following. A_3 and A_4 encode their message using $\sigma^i_{A_3}$ and $\sigma^j_{A_4}$, and send their particles to B_1 and B_2, respectively. B_1 and B_2 then decode the message by performing a joint measurement on all four particles in the $\{|\chi^{ij}\rangle_{A_3A_4B_1B_2}\}$ basis. It is easy to see that $\mathcal{D}_0$ works perfectly, enabling A_3 and A_4 to communicate 4 bits of classical information with B_1B_2 by sending in total 2 particles. This is impossible with a four-party GHZ or W state. However, we note that, whereas A_3 and A_4 may encode their message locally and hence independently, B_1 and B_2 are compelled to read the message together. One is not able to do it without the other's presence and cooperation. This is in contrast to a straightforward extension of the original dense coding scheme of Bennett *et al.* [16] to one involving two Bell states shared between $A_3(A_4)$ and $B_1(B_2)$, where B_1 and B_2 can individually read the respective message from A_3 and A_4. We denote this scheme by $\mathcal{S}_0$. This difference between $\mathcal{D}_0$ and $\mathcal{S}_0$ lies in the maximal entanglement between A_3B_1 and A_4B_2. In terms of the numbers of particles sent and the amount of classical information communicated, both $\mathcal{D}_0$ and $\mathcal{S}_0$ are exactly the same:

$$4 = \log_2 2^4 = 4\log_2 2 = 2\log_2 2 + 2\log_2 2 = 2 + 2. \quad (25)$$

An immediate example of a situation where $\mathcal{D}_0$ could have an advantage over $\mathcal{S}_0$ is the following: A_3 and A_4 wish to send some message to both B_1 and B_2, which they must both read at the same time together regardless of whether A_3 or A_4's particle reaches B_1 or B_2 first. We note that $\mathcal{D}_0$ works equally well between A_3B_1 and A_4B_2, but not between A_3B_2 and A_4B_1 because the entanglement between them is not maximal. In fact, from Eq. (19), we see that only

$$3 = \log_2 2^3 = 3\log_2 2 = \log_2 2 + 2\log_2 2 = 1 + 2 \quad (26)$$

bits of information can be transferred, if A_3 cooperate with B_2 by encoding only her qubit with either σ^0 or σ^3. In this case, A_4B_1 decode by measuring in the basis, Eq. (19), for an eight-dimensional subspace. This is consistent with Eq. (22).

In conclusion, we have shown that faithful teleportation of an arbitrary two-qubit state and dense coding are possible with $|\chi^{00}\rangle$. These can similarly be achieved using two Bell pairs. However, by construction, this state is different from a pair of Bell states. It is a genuine four-partite entangled state, which has properties that differ from those of four-party GHZ and W states. It could play an analogous role to $|\Psi^0_{\mathrm{Bell}}\rangle$ in the theory of multipartite entanglement.

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