

## Fracture Surfaces as Multiscaling Graphs

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Fracture paths in quasi-two-dimensional (2D) media (e.g., thin layers of materials or paper) are analyzed as self-affine graphs  $h(x)$  of height  $h$  as a function of length  $x$ . We show that these are multiscaling, in the sense that  $n$ th order moments of the height fluctuations across any distance  $\ell$  scale with a characteristic exponent that depends nonlinearly on the order of the moment. Having demonstrated this, one rules out a widely held conjecture that fracture in 2D belongs to the universality class of directed polymers in random media. In fact, 2D fracture does not belong to any of the known kinetic roughening models. The presence of multiscaling offers a stringent test for any theoretical model; we show that a recently introduced model of quasistatic fracture passes this test.

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*Background.*—The pioneering experimental work described in Ref. [1] drew attention to the fact that fracture surfaces are graphs in  $2 + 1$  ( $1 + 1$ ) dimensions when the broken sample is three-dimensional (two-dimensional). This initial insight was followed up by a considerable number of works [2,3] that focused on the scaling properties of such graphs under affine transformations. In  $1 + 1$  dimensions, one denotes the graph as  $h(x)$  and considers the structure function  $S_2(\ell)$ ,

$$S_2(\ell) \equiv \langle |h(x + \ell) - h(x)|^2 \rangle, \quad (1)$$

where angular brackets denote an average over all  $x$ . The statement is then that this function is a homogeneous function of its arguments,

$$S_2(\lambda\ell) \sim \lambda^{\zeta_2} S_2(\ell). \quad (2)$$

Close attention was paid to the numerical value of the scaling exponent  $\zeta_2/2$  as observed in rupture lines in quasi-two-dimensional materials (and in some numerical simulations) [4–7] with the hope of assigning to fracture a “universality class” of one of the well studied models of  $1 + 1$ -dimensional kinetic roughening models [8,9]. Indeed, since in many such  $1 + 1$  measurements the exponent  $\zeta_2/2$  was numerically close to 0.67, and also since it was proposed that fracture can be considered as a global minimization problem, some authors accepted a view that rupture lines map onto the model of directed polymers in random media [4,7–9]. The latter model is consistent with an exactly soluble exponent  $\zeta_2/2 = 2/3$  [9]. In this Letter, we show that this is not the case; moreover, we advance reasons to believe that rupture lines do not map on any of the known  $1 + 1$  kinetic roughening models but call for fresh thinking with new models in mind. One such new model will be shown to be a good candidate for this type of physics.

*Approach.*—We first argue in this Letter that the scaling properties of fracture lines are significantly richer than

what can be read from Eq. (2). To this aim, we consider the whole distribution function for height fluctuations  $P(h(x + \ell) - h(x))$  and the associated higher order structure function  $S_n(\ell)$

$$S_n(\ell) \equiv \langle |h(x + \ell) - h(x)|^n \rangle. \quad (3)$$

For rupture lines, the higher order structure functions come each with its own exponent

$$S_n(\lambda\ell) \sim \lambda^{\zeta_n} S_n(\ell), \quad (4)$$

where the scaling exponents  $\zeta_n$  are not simply related to the exponent  $\zeta_2$  (i.e.,  $\zeta_n \neq n\zeta_2/2$ ). In contrast, in directed polymers in random media the higher order structure functions bring in no new information, and there  $\zeta_n = n\zeta_2/2$  [10].

*Experimental example.*—An example that provides us with information of sufficient accuracy to establish the multiscaling characteristics is rupture lines in paper. The data acquired by Santucci *et al.* [11] was obtained in experiments where centrally notched sheets of fax paper were fractured by a standard tensile testing machine. Four resulting crack profiles  $h(x)$  were digitized. Each digitization contained a few thousand points, where care was taken to ensure that the smallest separation between points in  $x$  is larger than the typical fiber width; this is important to avoid the artificial introduction of overhangs that destroy the graph property.

Denoting  $\Delta h(\ell) \equiv h(x + \ell) - h(x)$ , we analyzed the data by boxing  $\ell$  in logarithmic boxes, accumulating the data between  $10^0$  and  $10^{0.25}$  (the smallest box) and between  $10^{2.25}$  and  $10^{2.5}$  (the largest box). The  $m$ th box was considered as representing data for  $\ell = 10^{m \times 0.25}$ . On the basis of this boxing, we constructed the probability distribution function (PDF)  $P(\Delta h(\ell))$  by combining data from all four samples. Samples that exhibit marked trends (probably due to the finite size of the sample) were detrended by subtracting the mean from each distribution. The computed

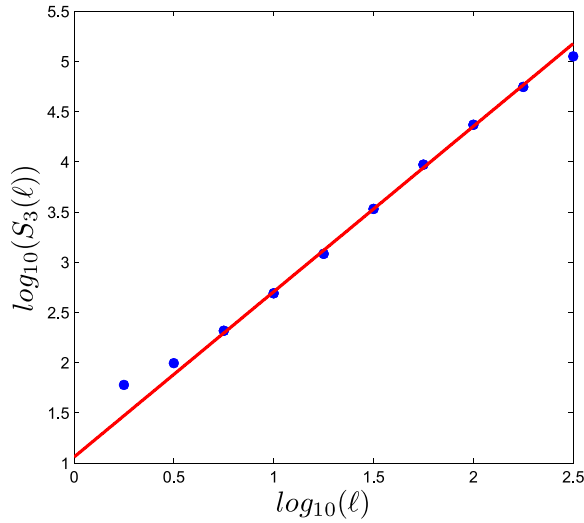


FIG. 1 (color online). A log-log plot of  $S_3(\ell)$  as a function of  $\ell$ . The linear fit corresponds to a typical scaling range of about 1.5 orders of magnitude, with a slope of  $\zeta_3 \approx 1.65$ .

PDF's were then used to compute the moments (3), and these in turn, once presented as log-log plots, yield the scaling exponents  $\zeta_n$ . Such a typical log-log plot is shown in Fig. 1, exhibiting a typical scaling range of about 1.5 orders of magnitude. Higher order structure functions with  $n \leq 8$  show equally convincing scaling ranges. The resulting values of the scaling exponents  $\zeta_n$  are shown in Fig. 2.

As a function of  $n$ , these numbers can be fitted to the quadratic function  $\zeta_n = nH - n^2\lambda$ , with  $H = 0.64 \pm 0.02$  and  $\lambda = 0.026 \pm 0.002$  (a linear plot  $n\zeta_1$  is added for reference). The error bars quoted here reflect both the

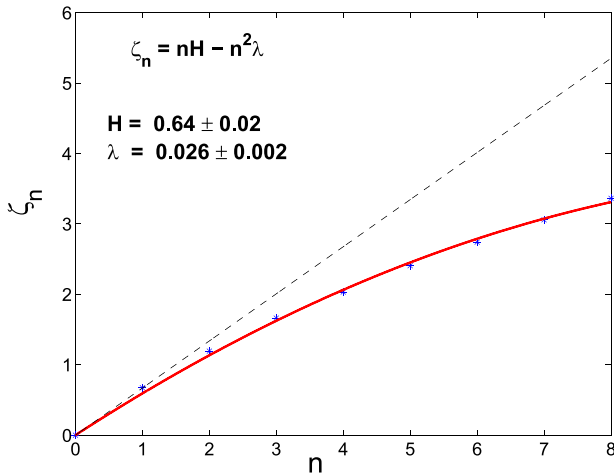


FIG. 2 (color online). The spectrum  $\zeta_n$  as a function of the moment order  $n$  for rupture lines in paper. The function is fitted to the form  $\zeta_n = nH - n^2\lambda$ , and the parameters  $H$  and  $\lambda$  are given. The errors in the estimation of these parameters reflect both the variance between different samples and the fit quality. The linear plot  $n\zeta_1$  is added to stress the nonlinear nature of  $\zeta_n$ .

variance between different samples and the fit quality. One reliable way to estimate what is the highest moment that the data can faithfully reproduce is to plot the integrand in the definition of the  $n$ th moment [12]:

$$\int |\Delta h(\ell)|^n P(\Delta h(\ell)) d\Delta h(\ell). \quad (5)$$

The integrand has to decay fast enough and be sufficiently smooth to be trusted. In our case, the discrete version of this integrand could not be trusted for all  $n > 8$ . On the other hand, the convergence of the 8th order moment is demonstrated in Fig. 3.

The point to stress is that the scaling exponents  $\zeta_n$  depend nonlinearly on  $n$  [3]; for the range of  $n$  values for which the moments converge, the exponents can be fitted to a quadratic function. It is well known from other areas of nonlinear physics, and turbulence in particular [13–15], that such phenomena of multiscaling are associated with PDF's on different scales  $\ell$  that cannot be collapsed by simple rescaling. In other words, in the absence of multiscaling, there exists a single scaling exponent  $H$  with the help of which one can rescale the PDF's according to

$$P(\Delta h(\ell)) \sim \ell^{-H} f\left(\frac{\Delta h(\ell)}{\ell^H}\right), \quad (6)$$

where  $f(\cdot)$  is a scaling function. In our case, such rescaling does not result in data collapse. In Fig. 4, the natural logarithm of  $P(\Delta h(\ell))\ell^H$  is plotted as a function of  $\Delta h(\ell)/\ell^H$  for  $H = 0.64$ . Indeed, the data do not collapse onto a single curve. The fat tails of the probability distribution functions at smaller scales are typical to multiscaling situations.

The known kinetic roughening models, and, in particular, directed polymers in random media, do not exhibit the multiscaling spectrum we have found, and thus we can infer that the scaling properties of rupture lines in 1 + 1

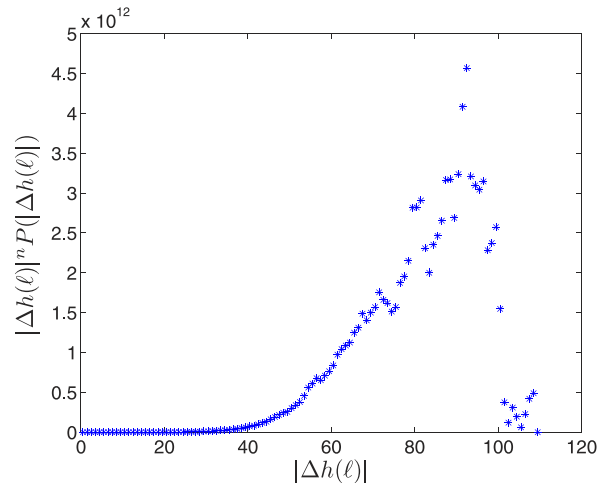


FIG. 3 (color online). An example of the convergence of the integral in Eq. (5) for  $\ell = 10^{2.25}$  and  $n = 8$ .

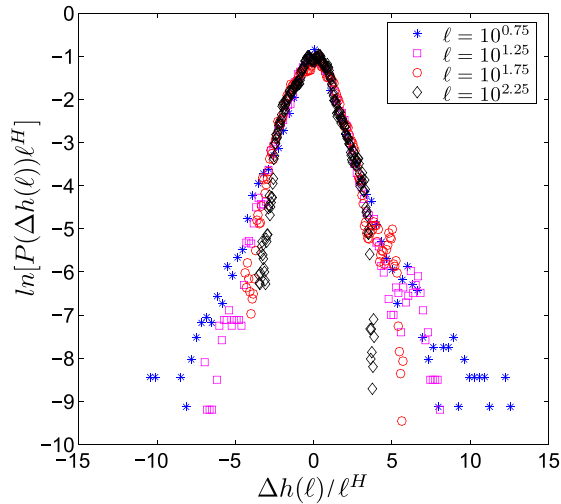


FIG. 4 (color online). The natural logarithm of  $P(\Delta h(\ell))\ell^H$  as a function of  $\Delta h(\ell)/\ell^H$  for  $H = 0.64$ . The legend gives the scale  $\ell$  for each plot.

dimension do not fall in the universality class of any one of the former. We propose that the long range elastic interaction that is typical to fracture is the origin of the positive correlations resulting in a nontrivial exponent (note that the quasistatic fracture problem involves the solution of the bi-Laplacian equation). To appreciate this important difference from standard kinetic roughening models in  $1 + 1$  dimensions, we turn now to a similar analysis of a theoretical model.

*Theoretical example.*—A recently published work [16,17] described a crack growth model aiming at understanding quasistatic fracture via void formation and coalescence. The model consisted of two components: (i) the exact solution of the elasticity problem in the presence of an arbitrarily shaped crack and (ii) a growth law in which the evolution of the crack is controlled by the nucleation of voids at a *finite distance*  $R$  ahead of the crack tip. The first component involved the adaptation of the method of iterated conformal maps to the field of elasticity in complex geometrical domains [18]. The second component incorporates an important physical insight about the role of a typical *length scale*  $R$  in the problem. This length scale determines where voids nucleate, making the crack growth a succession of rapid growth steps interrupted by slower void nucleation events. This is a crucial factor that enables the crack to develop positive correlations (i.e., a roughness exponent such that  $H > 1/2$ ) [19]. The succession of growth steps introduce variations in the crack geometry on the scale of the step size  $R$ . At a distance  $R$  ahead of the crack tip, these geometric irregularities change the solution of the stress field and mediate positive correlations. When the growth takes place right at the crack tip [20,21], the crack appears locally straight; the geometric irregularities are effectively screened and no positive correlations appear; i.e., one obtains  $H = 1/2$ . The origin of the finite

length scale that is involved in the crack growth process can be the near tip nonlinear physics (for example, plasticity; see [16,17]) or the scale of the quenched disorder in the system. The existence of a finite scale ahead of the tip appears crucial for the development of a nontrivial roughness exponent.

In Refs. [16,17] it was shown that this particular growth model generates rupture lines with  $\zeta_2/2 \approx 0.66$ . It is thus interesting to examine whether this model exhibits the same kind of multiscaling that was found in the experimental example above. Unfortunately, due to the significant computational cost of the iterated conformal maps technique, the numerical investigation of the growth model had a limited number of realizations of a few hundred growth steps. Because of the relative paucity of data, the structure functions defined in Eq. (3) would not converge well enough to provide reliable exponents. Instead, we use the max-min method which was proved reliable for the range of exponents under study and for the typical length of our cracks [22]. Thus, we define

$$\tilde{S}_n(\ell) \equiv \langle |\max\{h(\tilde{x})\}_{x < \tilde{x} < x + \ell} - \min\{h(\tilde{x})\}_{x < \tilde{x} < x + \ell}|^n \rangle. \quad (7)$$

The scaling exponents  $\zeta_n$  are defined in an analogous way to Eq. (4) by

$$\tilde{S}_n(\lambda\ell) \sim \lambda^{\zeta_n} \tilde{S}_n(\ell). \quad (8)$$

It should be stressed that for the experimental data this way of measurement resulted in scaling exponents within the same error bars as those shown in Fig. 2.

The resulting exponents  $\zeta_n$  for the cracks generated by this model are shown in Fig. 5. The integrand test that converged up to  $n = 8$  for the experimental data begins to fail for  $n \geq 6$  for the model data. For the converged moments, one again fits a quadratic function, with  $H = 0.66 \pm 0.02$  and  $\lambda = 0.023 \pm 0.002$ . The errors in the estimation of these parameters reflect both the variance between different realizations and the fit quality. The  $n$  dependence of the exponents  $\zeta_n$  and the values of the fitting parameters are in agreement with the experimental ones. Since there is nothing in the model that is specific for the physics of paper, it appears that multiscaling is a generic property of the fracture process, at least in  $1 + 1$  dimensions.

*Summary.*—We examined an experimental example of rupture lines and a theoretical model of rupture in  $1 + 1$  dimensions and showed that both exhibit graphs whose scaling properties appear similar, falling in a different class compared to standard models such as directed polymers in random media. The scaling exponents of higher order structure functions associated with graphs of rupture lines depend nonlinearly on the order of the moments, while in directed polymers this is not the case. In fact, it appears that none of the standard known growth models in  $1 + 1$  dimensions is in the same class as the rupture lines discussed

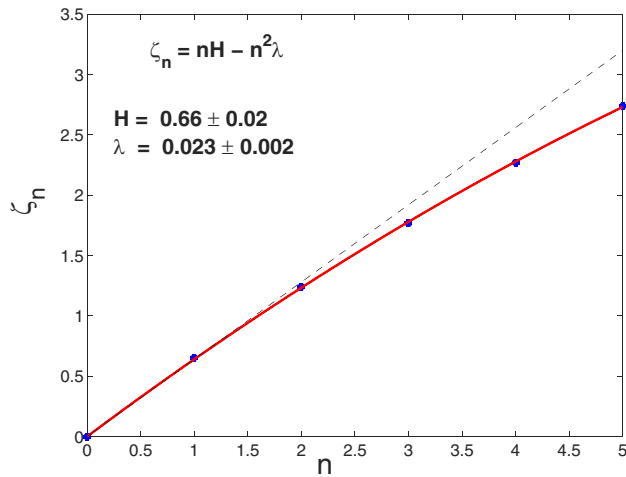


FIG. 5 (color online). The spectrum  $\zeta_n$  as a function of the moment order  $n$  for rupture lines in the model of Refs. [16,17]. The function is fitted to the form  $\zeta_n = nH - n^2\lambda$ , and the parameters  $H$  and  $\lambda$  are given. The errors in the estimation of these parameters reflect both the variance between different realizations and fit quality. The linear plot  $n\zeta_1$  is added for comparison.

here. The analysis of the theoretical model appears encouraging, in the sense that it captures the multiscaling characteristic of the experimental example. We reiterate our proposition that the crucial aspect of the theoretical model that is responsible for the nontrivial scaling behavior is the existence of a finite scale  $R$  that does not exist in pure elasticity theory or in models that treat the crack dynamics as a continuous process.

Further experimental work is necessary to provide further examples of rupture lines in  $1 + 1$  dimensions. With more examples, one would be able to determine the degree of universality of the scaling exponents, the dependence of the multiscaling exponents on issues such as the isotropy of the cracking medium, the presence of plasticity, and the protocol of the rupture experiments.

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