Revealing the Building Blocks of Spatiotemporal Chaos: Deviations from Extensivity

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We have performed high-precision computational studies of the fractal dimension as a function of system length for spatiotemporal chaotic states of the one-dimensional complex Ginzburg-Landau equation. Our data show deviations from extensivity on a length scale consistent with the chaotic length scale, indicating that this spatiotemporal chaotic system is composed of weakly interacting building blocks, each containing about 2 degrees of freedom. Our results also suggest an explanation of some of the ''windows of periodicity'' found in spatiotemporal systems of moderate size.

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Researchers seeking to understand the macroscopic properties of systems composed of a large numbers of objects (e.g., atoms, magnetic spins, stock traders, etc.) often employ a statistical approach in which the interactions between the objects are treated probabilistically, avoiding the necessity of knowing the details of each individual interaction. Equilibrium statistical mechanics, the application of this approach to systems in thermodynamic equilibrium, has led to an understanding of the phases of matter, the transitions between these phases, and the deep property of universality, which explains why systems that are physically quite distinct (for example, gases and magnets) behave identically in fundamental ways near phase transitions. Although researchers have extended equilibrium statistical mechanics to systems only slightly perturbed away from equilibrium, scientists have largely been stymied in their efforts to develop a similar approach for broad classes of systems far from equilibrium [1–3]. A particularly intriguing set of far-from-equilibrium systems exhibit the phenomenon of spatiotemporal chaos (STC), which is characterized by a chaotic dynamics that persists indefinitely and by spatial disorder often punctuated by topological defects or patches of uncorrelated regions [1–3]. Large, deterministic systems as diverse as fibrillating heart tissue [4], chemical reaction-diffusion systems [5], convecting fluid layers [6], and colonies of microorganisms [7] display this remarkable behavior. Often a particular spatiotemporal chaotic system will behave qualitatively differently for different system parameters, and the transitions between these chaotic ''phases'' closely resemble the phase transitions in equilibrium systems [3,5,8–12]. For one such system, researchers have even demonstrated that the long wavelength behavior is indistinguishable from an equilibrium system of the Ising universality class [13]. Despite these provocative findings from experiments and simulations, little progress has been made toward developing a predictive theory of these systems.

To develop a statistical mechanics of STC, an understanding of the effective degrees of freedom in the system is almost certainly necessary. For equilibrium systems it is usually intuitively clear which degrees of freedom determine the macroscopic behavior. For example, in a study of the behavior of a gas of argon atoms, the atoms, and not the nucleons or the quarks comprising the nucleons, are the fundamental degrees of freedom. The faster dynamics of the nucleons and quarks within the nucleus is slaved to the slower dynamics of the atom as a whole such that the macroscopic behavior of the gas is determined by the interactions of the atoms as ''fundamental'' units. In contrast, STC systems usually contain a broad spectrum of length scales and time scales, making it unclear how to identify the appropriate fundamental degrees of freedom underlying the macroscopic behavior; however, the fractal dimension D can be considered a measure of the average number of independent degrees of freedom necessary to characterize the state of the system [14]. For large STC systems, D has been found to be extensive [10,15–18], meaning that it grows linearly with the volume of the system L^d, where L is a characteristic length and d is the dimensionality of the system. Ruelle has argued [19] that the extensivity of D arises in STC due to spatial disorder: since distant regions are uncorrelated, their dynamics is also uncorrelated, and thus each region contributes independently to D. Cross and Hohenberg [1] make this more explicit by defining a natural chaotic length scale,

$$\xi_\delta \equiv \left(\frac{D}{L^d}\right)^{-1/d}, \tag{1}$$

for large L, such that a volume of size ξ_δ^d contains on average 1 degree of freedom. The chaotic length scale ξ_δ is typically small compared to length scales that characterize the macroscopic state of the system [10,16–18,20], and, for two different systems, researchers have found a physical quantity with spatial correlations of a length scale proportional to ξ_δ [10,17,21], providing hope that each degree of freedom might be directly associated with a region of the system of volume ξ_δ^d. Further bolstering this case is a study of a two-dimensional STC system containing topological defects, in which it was shown that each defect could be associated with approximately 2 degrees of freedom [22].

Two research groups [23,24], however, have cast doubt on the interpretation of ξ_δ as a direct measure of the extent of individual degrees of freedom, and, indeed, whether the degrees of freedom are even localized. If the underlying building blocks of STC are objects of size about ξ_δ^d, it is likely that a careful study of D as a function of system volume L^d would show structure on the scale of ξ_δ^d, while still maintaining an extensive growth on average. For example, one possibility would be a step structure with D completely quantized—increasing by 1 each time the system is increased in volume by enough to add exactly one more building block. A smoother increase is also possible, since, for example, the system could contain different numbers of building blocks at different points in time, with the blocks "stretching" or "compressing" a little to match the system size. The system might jump back and forth (perhaps in a process analogous to thermal sampling) between states containing a number of building blocks close to, but not exactly, L^d/ξ_δ^d. In such a situation, there would likely still be structure at the scale of ξ_δ^d, but averaging would reduce the magnitude of the deviations from extensivity in comparison to the completely quantized scenario. Researchers tested this idea on two different STC systems and found that D was, instead, "microextensive" [23,24], meaning that for large systems the dimension grew strictly linearly (to the precisions of their studies) even for very small changes in the length of the system. These findings significantly weakened the idea that STC could be treated as a collection of D spatially distinct degrees of freedom.

Here we report high-precision studies of a prototypical spatiotemporal chaotic system showing, in contrast to these earlier works, clear violations of microextensivity. These violations occur on a length scale consistent with ξ_δ over a range of system parameters, providing solid evidence that STC is composed of interacting chaotic building blocks of size ξ_δ. The magnitudes of the violations are small, perhaps implying that the effective temperature is large (or, alternatively, that the forces between the building blocks are weak), and we argue that the form of the violations may reveal the nature of the symmetry of the effective potentials (or forces). Furthermore, the nature of the violations suggests an explanation of the "windows of periodicity" found in STC systems of moderate size.

To investigate the possibility of deviations from microextensivity, we studied numerical solutions of the complex Ginzburg-Landau equation in one spatial dimension ($d = 1$):

$$\partial_t A = A + (1 + ic_1)\partial_x^2 A - (1 - ic_3)|A|^2 A, \qquad (2)$$

where $A(x, t)$ is a complex-valued field on a periodic spatial interval of length L, and c_1 and c_3 are real constants. Equation (2) describes any system near the onset of a Hopf bifurcation from a stationary, homogeneous state to an oscillatory state [1], including a number of systems that have been studied experimentally [1,25–27]. This equation has received much attention because it is a relatively simple continuum system that exhibits a variety of behaviors for different values of the parameters c_1 and c_3 [8,10,27]. For our studies, we maintained $c_1 = 3.5$ and varied c_3 from 0.85 to 1.20. For this range of parameters, the system exhibits defect chaos, and the equal-time, two-point correlation length ξ_2 of the field A, a measure of the macroscopic coherence, varies by a factor of about 20 whereas ξ_δ varies by a factor of about 1.5.

We used a pseudospectral method with time splitting of the operator [28] to numerically integrate Eq. (2). We ensured that, to within the error bars on our measurements, our results did not change with finer spatial or temporal resolution, with longer integration times, or even with changes in the components of the integration scheme. Each data point represents an ensemble average over at least 192 different initial conditions.

We calculated a particular fractal dimension, the Lyapunov dimension D, in terms of the spectrum of Lyapunov exponents [29]. These exponents characterize the temporal evolution of solutions of Eq. (2) that differ by only infinitesimal amounts. We calculated the exponents using a computationally expensive technique [29] that involves integrating not only Eq. (2) but also a large number of copies of the linearization of Eq. (2) about a solution $A(x, t)$ [30]. If the exponents are summed in order from largest to smallest, D is the number of exponents needed to reach a sum of zero. Because of the discrete nature of the Lyapunov spectrum, the sums, in general, are never exactly zero, and the value of D is determined by interpolation using the 5 sums with values closest to zero [31].

Figure 1(a) shows D as a function of the system length L for $c_3 = 0.95$. D grows approximately linearly with L, showing the extensive nature of the dimension. Oscillations about linear, microextensive growth can be seen for small values of L. To highlight these oscillations and to reveal the oscillations of smaller magnitudes, we show in Fig. 1(b) the relative deviation from microextensive behavior, $\Delta D = (D - D_{\text{ext}})/D_{\text{ext}}$, as a function of L. The clear oscillations in Fig. 1(b) lead to the central conclusion of this Letter. Fitting the region $44 \leq L \leq 68$ (about 3 wavelengths) to a cosine with decaying amplitude yields an oscillation wavelength of $\xi_o = 8.54$ spatial units, which is approximately twice the chaos length scale $\xi_\delta = 4.46$ determined using Eq. (1) at large L. So, the chaotic building blocks for this state are about 8.5 units in length and each contains approximately 2 degrees of freedom. The wavelength of the oscillations appears to be constant as L is varied, but the amplitude falls off rapidly with increasing L. This decrease in magnitude is expected because for larger values of L the effect of adding additional system length is spread across a larger number of blocks.

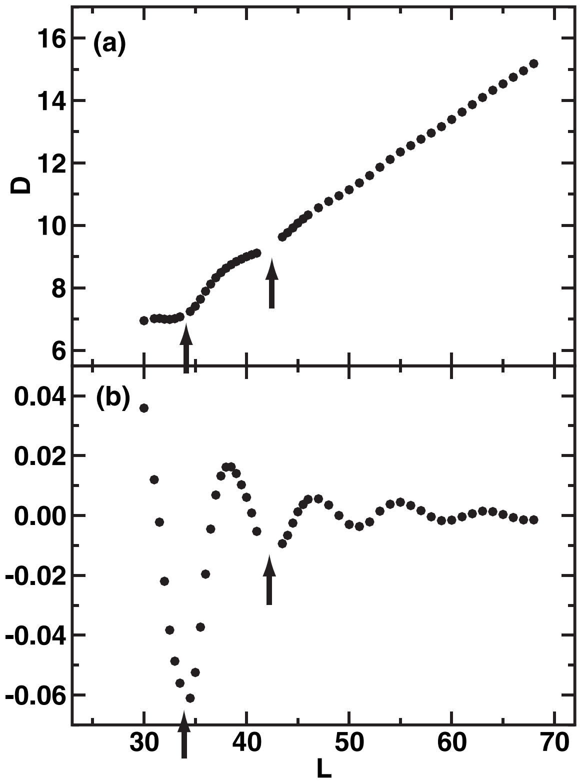

FIG. 1. (a) Dimension D as a function of system length L for Eq. (2) with $(c_1, c_3) = (3.5, 0.95)$. (b) Relative deviations $\Delta D = (D - D_{\text{ext}})/D_{\text{ext}}$ from microextensive behavior as a function of L, with $D_{\text{ext}}(L) = \delta L$ and δ determined at large L. Oscillations of length scale $\xi_o \approx 8.5$ signal the presence of chaotic building blocks. The arrows indicate missing values of L for which nonchaotic behavior is found. Error bars for D, determined from the standard deviation of 200 independent measurements, are typically smaller than 0.002. Each of the 200 measurements is an average of many samples during a time period of at least 75 000 time units following a transient of 80 000 time units, with time step $dt = 0.05$ and at least 2 spatial modes per spatial unit.

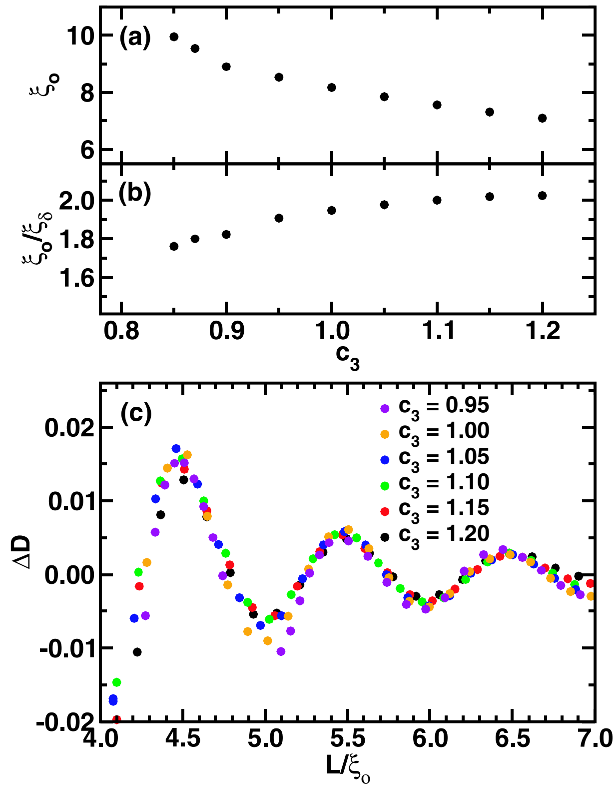

FIG. 2 (color online). (a) Oscillation length scale ξ_o as a function of c_3 for fixed $c_1 = 3.5$ in Eq. (2). Values of ξ_o were determined by fitting a cosine with decaying amplitude to $\Delta D(L)$. A span of at least 3 wavelengths was used for $c_3 \geq 0.90$. (b) Ratio ξ_o/ξ_δ for the same values of c_3. ξ_δ is determined using Eq. (1) at large L. The data indicate that each building block contains about 2 degrees of freedom. (c) Relative deviation ΔD as a function of building block number, L/ξ_o, for 6 values of c_3. The collapse shows that the deviations depend only on the number of building blocks. Minima occur for integer numbers of building blocks. The data near the minimum at $L/\xi_o = 5$ for $c_3 = 0.95$ and $c_3 = 1.00$ are influenced by a window of periodicity.

We speculate that the small magnitude of the oscillations in Fig. 1(b) indicates that the effective temperature of the system is high (or, alternatively, that the effective forces between the building blocks are weak). If the decrease in magnitude is factored out, the oscillations appear to be symmetric, perhaps indicating that the intrablock force is symmetric with regard to "stretching" and "compression" and thus only contains even powers of the block separation. With further investigation, the exact form of the effective intrablock force might be deduced from the shape of the deviations from microextensivity.

Figure 2(a) summarizes our measurements of ξ_o for several values of c_3. As we increase the value of c_3, the value of ξ_o decreases. In Fig. 2(b), we show the ratio of ξ_o to ξ_δ for each value of c_3. The ratio is approximately 2 across this range of parameters, so the building blocks contain about 2 degrees of freedom. We note that over this same range of parameters, the correlation length ξ_2 changes by a factor of about 20 [32].

Figure 2(c) shows ΔD as a function of the number of building blocks that fit in the system, L/ξ_o, for 6 values of $c_3 \geq 0.95$. (Smaller values of c_3 are subject to strong finite-size effects for this range of L/ξ_o.) The collapse of the data onto a single cosine function with decaying amplitude indicates that the deviations depend only on the number of building blocks in the system. Intriguingly, the states closer to fitting an exact number of building blocks are less chaotic (in terms of dimension per length). Perhaps the additional dynamics arising from the frustration of mismatched lengths yields a small contribution to D for the states further from these minima.

Our finding that STC is composed of spatially localized building blocks also suggests a possible explanation for at least some of the mysterious "windows of periodicity" observed in spatiotemporal chaotic systems. It has been well documented [33] that periodic (or other nonchaotic) behavior can be found in moderately sized systems, even when both larger and smaller systems exhibit STC. To date, the appearance of these windows of nonchaotic behavior has not been understood. The arrows at $L = 34.0$ and $L = 42.5$ in Fig. 1 indicate small ranges of values of L for which we observe nonchaotic behavior instead of STC. Similar behavior is observed for values of $c_3 \leq 0.95$. The windows occur at the minima in the oscillations—system lengths that fit exact numbers of building blocks. We speculate that these windows may be the result of the system finding perfect alignment of the blocks, whereas for larger integer multiples of ξ_o, the system never manages to align the blocks [perhaps in analogy to the way finite-size effects lead to early crystallization of a liquid or to the competition between spiral defect chaos and stationary states in convection systems of moderate size for which rolls are stable [6]]. For noninteger multiples of ξ_o, the frustration of the mismatched lengths prevents the alignment.

Our data for the complex Ginzburg-Landau equation strongly suggest that STC is composed of weakly interacting building blocks, each of which contains about 2 degrees of freedom. This finding provides hope that a statistical mechanics of STC might be built by considering these blocks and their interactions.

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[30] The data are somewhat sensitive to the frequency with which the linearized solutions are orthogonalized; however, over a range of frequencies, changes to this numerical parameter only appear to cause a small shift in the dimension measurements, not a change in the oscillation length scale.

[31] The usual linear approximation of the Kaplan-Yorke formula [29] is inadequate for our purposes and introduces an artificial oscillation; however, higher-order interpolations using 3, 4, or 5 points yield values of D that differ from each other typically by an amount smaller than the error bars in our calculations.

[32] The rapid increase in ξ_2 as c_3 is lowered effectively sets the lower bound for c_3 in our study since, when L is smaller than about $4\xi_2$, there are relatively large deviations due to finite-size effects other than those we are trying to measure. Since the magnitude of the fluctuations we are trying to isolate decreases quickly with L, it is currently impractical to go to lower values of c_3. However, even for $L < 4\xi_2$ signatures of the oscillations at length scale ξ_o are sometimes present. The data points for $c_3 = 0.85$ and $c_3 = 0.87$ were determined from these oscillations riding on the larger finite-size behavior and so are considerably less accurate than the data for $c_3 \geq 0.90$.

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