Diffusion-Induced Inhomogeneity in Globally Coupled Oscillators: Swing-By Mechanism

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It is shown that for random initial conditions, a large population of identical and sufficiently nonisochronous Stuart-Landau oscillators coupled globally and diffusively exhibits inhomogeneity in a resonant way as the diffusive coupling is intensified, where the diffusive coupling constant is real. A category of inhomogeneous (nonsynchronized) solutions is analytically shown to exist, which is either periodic or quasiperiodic.

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The role of diffusive coupling in nonlinear dynamics is quite interesting, because such coupling sometimes makes the system inhomogeneous rather than uniform, in contradiction with the familiar understanding of its effect. A well-known example is the Turing instability in diffusively coupled activator-inhibitor systems, where diffusion causes an instability of a spatially uniform state when the inhibitor diffuses sufficiently faster than the activator [1,2]. Another example is found in a neurophysiological context, in which synchronization of periodic firings of model neurons is broken by a diffusive coupling involving only one variable [3]. In this example, strong "nonisochronicity" (i.e., amplitude dependency of frequency) [4] due to the existence of a saddle near the limit cycle and the one-variable nature of the coupling are responsible for such a peculiar effect [5].

What is common with these examples of diffusion-induced inhomogeneity is that the diffusive coupling is not symmetric for the variables involved, i.e., there is a substantial difference between their diffusion constants or parameters playing a similar role. In fact, this feature causes at least in part the counterintuitive effects of diffusive coupling in the above examples. In contrast, in this Letter, we show that under some conditions, even perfectly symmetric diffusive coupling can induce inhomogeneity in globally coupled identical limit-cycle oscillators. A curious aspect of the phenomenon reported here is that the degree of inhomogeneity of the system exhibits a resonancelike behavior as the diffusive coupling is intensified. The mechanism giving rise to these results is elucidated below by investigating the relaxation of the system from random initial conditions. Hence, this work may be expected to provide a new clue as to how to control the coherence of real coupled-oscillator systems [1,2,4]. Simulation results of this work were obtained by means of the fourth order Runge-Kutta method with time step of 0.01.

We consider a large population of Stuart-Landau oscillators [6–10] as follows:

$$\dot{z}_j = z_j - (1 + ic_2)|z_j|^2 z_j + \frac{K}{N}\sum_{k=1}^{N}(z_k - z_j) \qquad (1)$$

for $j = 1, \ldots, N$, where the dot means differentiation with respect to time t, z_j is the complex amplitude of the jth oscillator, c_2 is a parameter to adjust the strength of nonisochronicity, and $K(>0)$ is the coupling strength. When uncoupled, all the oscillators in Eq. (1) are identical limit-cycle oscillators with an amplitude unity and a frequency $-c_2$. Note that the diffusive coupling in Eq. (1) is a scalar type, so that the inhomogeneity-generating mechanisms based on unequal diffusion constants will not work in the present system. We may therefore expect that any increase in K will either improve or maintain the homogeneity of the system. Figure 1 reveals that this naive expectation can be wrong. It shows the behavior of the standard deviation of the complex amplitudes defined by

$$\sigma = \langle(\overline{|z_j - \overline{z_j}|^2})^{1/2}\rangle, \qquad (2)$$

where the bar means an average over $1 \le j \le N$ and the

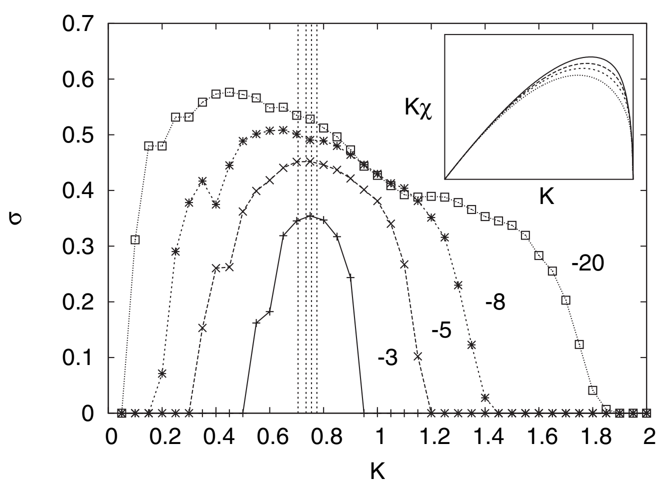

FIG. 1. The behavior of σ for $N = 4000$ averaged over ten realizations of the random initial condition explained in text. The numbers attached to the data are the values of c_2. The lines connecting the symbols are to guide the eye. The inset shows $K\chi$'s graphs in the range $0 \le K \le 1$, $0 \le K\chi \le 0.6$ for the same values of c_2, where its maximum values as well as its maximum points indicated by the vertical dotted lines in the main panel monotonically decrease for increasing $|c_2|$.

brackets stand for a time average. The data presented in Fig. 1 were obtained by averaging σ further over a number of random initial conditions taken in the range $-1 < \mathrm{Re}(z_j), \mathrm{Im}(z_j) < 1$. For small K, σ vanishes, implying that the system falls in perfect synchronization. However, as K is increased, σ eventually starts to increase with K, then becoming maximum, and finally decreasing towards zero. It should be noted that this resonancelike behavior of σ cannot be attributed to destabilization and restabilization of the synchronized state, since it can be shown to be always stable and K's increase simply strengthens its stability [8]. The simulation result therefore suggests that in the resonant regime, some nonsynchronized, inhomogeneous attractors stably coexist with the synchronized one in phase space and that by a certain mechanism, one of them is chosen for a random initial condition in such a way that on average, the inhomogeneity of the chosen attractor is resonantly enhanced as K is increased. Figure 2 shows a couple of examples of such inhomogeneous solutions [11].

We now discuss the mechanism of the appearance of such solutions for $N \gg 1$ by assuming that the nonisochronicity is sufficiently strong. We start by rewriting Eq. (1) as follows:

$$\dot{r}_j = (1 - K - r_j^2)r_j + KR\cos(\Theta - \theta_j), \tag{3}$$

$$\dot{\theta}_j = -c_2 r_j^2 + \frac{K}{N}\sum_{k=1}^{N}\frac{r_k}{r_j}\sin(\theta_k - \theta_j), \tag{4}$$

where $z_j \equiv r_j e^{i\theta_j}$ and $N^{-1}\sum_{k=1}^{N} z_k \equiv Re^{i\Theta}$ with $r_j, R \geq 0$. Now we suppose that as in the simulation, the initial values of the complex amplitudes are randomly distributed in a sufficiently large area which is symmetric about the origin. Then, the initial value of R, $R(0)$, should be as small as $1/\sqrt{N}$ and remain small for some time. Hence, as we can see from Eqs. (3) and (4), every oscillator will approach the same circle on the complex plane centered at the origin with a radius $\sqrt{1-K}$, keeping their phase values randomly distributed. Namely, the system will approach an incoherent state defined by $r_1 = \cdots = r_N = \sqrt{1-K}$ and $\sum_{k=1}^{N} e^{i\theta_j} = 0$ [12]. For convenience, the circle will hereafter be called the *incoherent circle* (IC). The incoherent state can be shown to be unstable in the present system [8,9], which fact underlies the following argument.

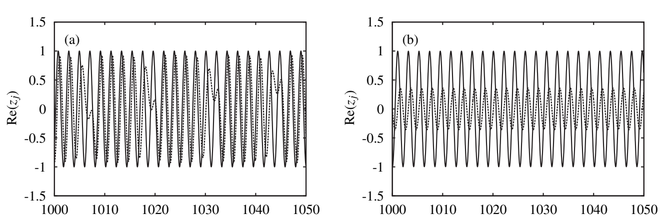

FIG. 2. Examples of desynchronized solutions for $c_2 = -3$, $N = 1000$. There are two synchronized sets of oscillators; the bigger (size $N - M$) and the smaller (size M) are drawn by real and broken curves, respectively. (a) $K = 0.51$ (quasiperiodic, $M = 4$). (b) $K = 0.94$ (periodic, $M = 9$).

Note that as the oscillators approach the IC, Eq. (4) tends to reduce to the Kuramoto model [2] for identical phase oscillators as

$$\dot{\theta}_j = -c_2(1 - K) + \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_j), \tag{5}$$

which is known to synchronize for $K > 0$ unless the phase distribution is perfectly uniform, so that the oscillators will start to synchronize in phase and this in turn will make the mean-field magnitude R increase, because $R \propto Q$ near the incoherent state, where $Q \equiv |\frac{1}{N}\sum_{k=1}^{N} e^{i\theta_k}|$ is the order parameter of the oscillator phases. It is possible to show [13] that the proportionality constant $\chi \equiv R/Q$ equals the absolute value of $2(1-K)^{3/2}(1+ic_2)/\{2(1-K)\times(1+ic_2) - K + \lambda\}$, where λ is a stability eigenvalue of the incoherent state with the largest real part [8]. As a result, as Eq. (3) implies, the oscillators begin to depart the IC in such a way that r_j increases if $|\theta_j - \Theta| < \pi/2$ and decreases if otherwise. The assumed strong nonisochronicity should then split the whole population into, roughly speaking, two subpopulations characterized by large and small complex amplitudes, since their effective frequencies $-c_2 r_j^2$ are quite different. In this way, the system will evolve to an inhomogeneous state. This scenario is verified in Fig. 3, where the temporal behavior of r_j, θ_j, R, and Q from $t = 0$ is exemplified. It is evident that just around when the oscillators are on average closest to the IC, phase order begins to emerge, leading to the increase of R and the oscillators accordingly depart from the circle and then break up into two groups as predicted above.

The resonancelike behavior of σ may be heuristically explained as follows: Near the incoherent state, the coefficient KR of the phase-dependent term of Eq. (3) may be replaced by $K\chi Q$, so that the prefactor of Q, $K\chi$, is a measure of the strength of the inhomogeneity-triggering effect of the synchronization in phase. The dispersion σ will therefore have a maximum point near that of $K\chi$ (see the inset of Fig. 1). Moreover, in the neighborhoods of $K = 0$ and 1, this prefactor is small, so that no desynchronization will occur there. This picture explains the overall behavior of σ fairly well at least for $|c_2|$ not too large (see the data for $c_2 = -3$ and -5 in Fig. 1). For $|c_2|$ very large, synchronization in phase is destroyed almost as soon as the oscillators leave the IC, making the value of R rapidly decrease back towards zero. As a result, the oscillators approach the IC again and the whole process is repeated with R showing large-amplitude oscillations [13]. In this complicated regime, the above argument is no longer feasible and in fact, there are substantial discrepancies between the peak points of σ and those of $K\chi$, as seen in Fig. 1. It should also be noted that for $|c_2|$ large, the

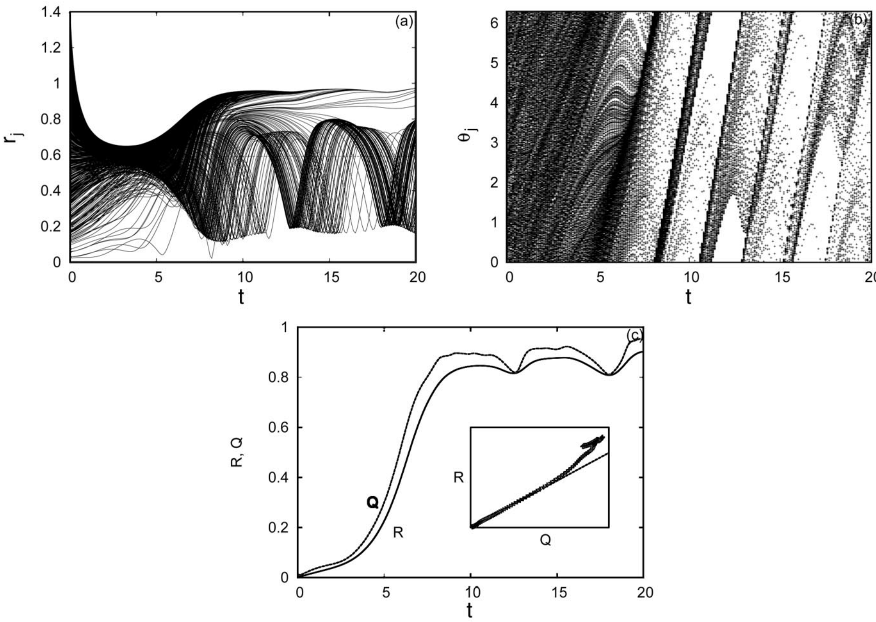

FIG. 3. Relaxation from a random initial condition for $c_2 = -3$, $K = 0.65$, $N = 1000$. (a) $r_j \equiv |z_j|$, where the horizontal broken line shows the radius of the incoherent circle. (b) $\theta_j \equiv \mathrm{Arg}(z_j)$. (c) Order parameters R and Q. The inset shows R vs Q over the ranges $0 < Q$, $R < 1$, where the dotted line shows $R = \chi Q$.

interval of K with $\sigma > 0$ penetrates into the region $K > 1$ in which the IC no longer exists. Here, the origin $z = 0$ plays a similar role as the IC [13]. We conjecture, however, that the inhomogeneity observed for $K > 1$ will disappear if the system size N is sufficiently large, because then the oscillators should approach one another too closely to remain nonsynchronized during the initial stage in which they move towards the origin. In fact, it was confirmed that for $|c_2|$ large, the upper end of the interval tends to recede as N is increased [14].

We now analytically show that in some region including the one where inhomogeneity emerges in simulation, there indeed exists a stable nonsynchronized solution. Our simulation indicates that a nonsynchronized solution is composed of one big cluster plus a number of minor clusters, where a "cluster" means a set of oscillators showing identical behavior, including those with only one element (see Fig. 4). Its composition not only depends on initial conditions, but also varies in time in some cases. Our strategy for such a complicated situation is to focus on the simplest category of inhomogeneous solutions as $z_j = z\ (1 \le j \le N-1)$, i.e., a two-cluster solution with the size ratio $(N-1)$:1. Noting that z_N's influence on the dynamics of z is negligible for N large, we obtain $z = e^{i\theta}$ with $\dot{\theta} = -c_2$ and

$$\dot{u} = (1 - K + ic_2)u - (1 + ic_2)|u|^2 u + K, \qquad (6)$$

where $u \equiv z_N e^{-i\theta}$. The last equation has a stable fixed point at $u = 1$, which corresponds to perfect synchronization of the population. Moreover, it can be shown that a pair of other fixed points exists for $K < (1 + c_2^2)/2(1 + \sqrt{1 + c_2^2}) \equiv K_{\mathrm{SN}}$; one of them is a saddle (say A), while the other (say B) bifurcates as a node and then changes to a focus. A linear stability analysis reveals that for $|c_2| < 1$, B remains unstable as K is decreased from

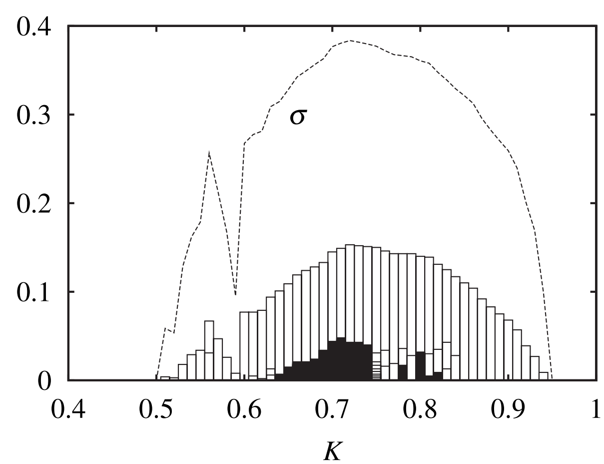

FIG. 4. The cluster structure of numerically found solutions for a random initial condition and $c_2 = -3$, $N = 1000$. Each segment of a vertical bar shows the size ratio of a cluster smaller than the main cluster, which is not shown here. These segments are placed in the order of their sizes, from top to bottom. Almost black regions correspond to the existence of a large number of small clusters. A cluster here is defined to be a collection of oscillators less than 10^{-5} apart from one oscillator at $t = 10\,000$. The behavior of σ is also shown.

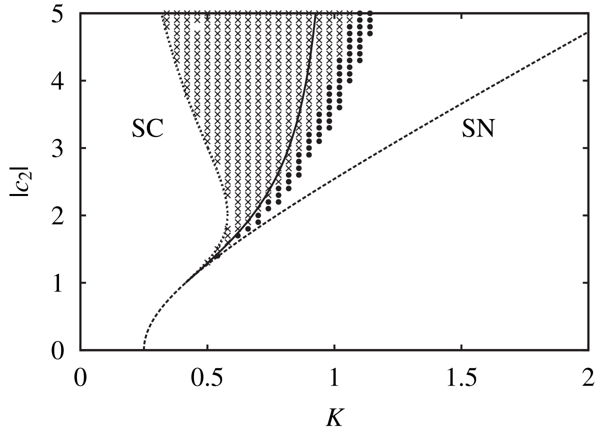

FIG. 5. The phase diagram based on the approximation developed in text, where the rightmost, middle, and leftmost curves show $K = K_{SN}, K_H, K_{SC}$, respectively. Note that the diagram does not depend on the sign of c_2. The symbols show those points which lead to an inhomogeneous solution in simulation of Eq. (1) for a random initial condition, where the nature of the solution is periodic (solid circle) or nonperiodic (cross).

K_{SN} to zero, while for $|c_2| > 1$, it stays stable until K reaches K_H at which a supercritical Hopf bifurcation occurs, where K_H is given by $[-2 + \sqrt{4 + (1 + c_2^2)^2}]/(1 + c_2^2)$; a stable limit-cycle created this way grows until it suddenly disappears by colliding with the saddle A to form a saddle connection at some point, say $K = K_{SC}$. Thus, it turns out that for $|c_2| > 1$, the original system (1) has a stable inhomogeneous solution in the range $K_{SC} < K < K_{SN}$, which is periodic with the same period as the synchronized solution for $K_H < K < K_{SN}$, whereas quasiperiodic for $K_{SC} < K < K_H$.

The above analysis is applicable even if one big cluster is accompanied by more than one (say, M) other oscillators, provided that $M \ll N$. Our simulation results indicate that in such a case, which is typical near the borders of the nonsynchronized region (see Figs. 2 and 4), the outsiders synchronize to form another cluster, approximately obeying the same dynamics as for $M = 1$. A phase diagram based on the present approximation is presented in Fig. 5, where simulation results for $N = 1000$ are also displayed. As is seen, inhomogeneity is observed for $|c_2| > 1$, in between the curves of K_{SC} and K_{SN}. Moreover, it is noteworthy that the onset of inhomogeneity occurs very close to the curve of $K = K_{SC}$. These results demonstrate that at least some aspects of simulation results can be captured by the present approximation.

In summary, a new type of diffusion-induced inhomogeneity in globally coupled identical oscillators has been proposed, which originates from the "swing by" of oscillators via the incoherent circle. Although we have concentrated on the case of coupled Stuart-Landau equations in this Letter, it is easy to generalize our argument and in fact, we have found similar phenomena in populations of other periodic oscillators such as Rössler systems. In this Letter, we have also developed an approximation method to study the simplest category of nonsynchronized solutions [15] and demonstrated its effectiveness in, e.g., explaining the emergence of quasiperiodicity in some region [16]. Hopefully, all these results will lead to a new method to control the coherence of globally coupled oscillators which appear in a variety of disciplines of science and technology.

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[1] A. T. Winfree, *The Geometry of Biological Time* (Springer, New York, 2001).
[2] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer, Berlin, 1984); I. Z. Kiss, Y. Zhai, and J. L. Hudson, Science **296**, 1676 (2002).
[3] A. Sherman and J. Rinzel, Proc. Natl. Acad. Sci. U.S.A. **89**, 2471 (1992).
[4] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization—A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, England, 2001).
[5] S. K. Han, C. Kurrer, and Y. Kuramoto, Phys. Rev. Lett. **75**, 3190 (1995).
[6] M. Shiino and M. Frankowicz, Phys. Lett. A **136**, 103 (1989).
[7] P. C. Matthews, R. E. Mirollo, and S. H. Strogatz, Physica D (Amsterdam) **52**, 293 (1991).
[8] V. Hakim and W.-J. Rappel, Phys. Rev. A **46**, R7347 (1992).
[9] N. Nakagawa and Y. Kuramoto, Prog. Theor. Phys. **89**, 313 (1993).
[10] H. Daido and K. Nakanishi, Phys. Rev. Lett. **93**, 104101 (2004).
[11] Nonsynchronized solutions of Eq. (1) are also studied by M. Tachikawa in *Meetings of the Physical Society of Japan*, March and September 2005 (unpublished).
[12] In fact, it is easy to prove that if the set of initial values $z_1(0), \ldots, z_N(0)$ is perfectly symmetric with respect to the origin, then R remains to be exactly zero and hence the system converges to an incoherent state, regardless of the system size N.
[13] H. Daido and K. Nakanishi (to be published).
[14] We remark that even if $R(0)$ is not small, qualitatively the same behavior as in Fig. 1 can follow when $|c_2|$ is very large, because the strong nonisochronicity automatically drives the system into a state with $R(0)$ small, as long as the dispersion of $r_j(0)$ exists.
[15] A similar spirit of analysis may be found in T. Mizuguchi and M. Sano, Phys. Rev. Lett. **75**, 966 (1995). We thank T. Mizuguchi for calling our attention to this Letter.
[16] Quasiperiodicity is also observed in globally coupled Van der Pol oscillators [M. Rosenblum (private communication)].