Soliton Solutions to the Einstein Equations in Five Dimensions

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We present a new class of solutions in odd dimensions to Einstein's equations containing either a positive or a negative cosmological constant. These solutions resemble the even-dimensional Eguchi-Hanson–(anti)-de Sitter [(A)dS] metrics, with the added feature of having Lorentzian signatures. They provide an affirmative answer to the open question as to whether or not there exist solutions with a negative cosmological constant that asymptotically approach AdS_5/Γ but have less energy than AdS_5/Γ. We present evidence that these solutions are the lowest-energy states within their asymptotic class.

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Weakly coupled non-Abelian gauge theories on non-simply connected manifolds can have ground states of much lower energy than one might naively expect. For example, a $U(N)$ gauge theory on a torus of m dimensions whose typical length is L can have its lowest-energy states of order $1/(NL)$ (instead of $1/L$) if N of the fields are arranged to be periodic after traversing one circle N times [1]. By introducing such locally flat but globally nontrivial connections, the effective size of the compact space is thereby increased by a factor of N, correspondingly reducing the spectrum of states. A string-theoretic interpretation of this phenomenon is that of the low-energy excitations of one D-brane wrapped N times around a circle, where the $U(N)$ gauge theory describes the low-energy excitations of N D-branes wrapped on the torus. The N^2 open strings connecting distinct D-branes in the latter case become N multiply identified open strings on a circle of length NL in the former case, yielding a configuration with lower energy states.

The preceding construction can be generalized [2] to quotient spaces M/Γ, where Γ is any freely acting discrete group of isometries on the compact Riemannian manifold M, provided $N = n|\Gamma|$, with $|\Gamma|$ the number of elements in the group. By wrapping n branes $|\Gamma|$ times each around M, one obtains the same $U(N)$ gauge theory as if one had wrapped N branes once each around M/Γ.

In this context, the anti-de Sitter/conformal field theory (AdS/CFT) correspondence conjecture implies the existence of extra light states in a gauge theory formulated on a quotient space that can be regarded as the boundary of an asymptotically AdS spacetime. Specifically, the conjecture states that string theory on spacetimes that asymptotically approach $AdS_5 \times S^5$ is equivalent to a conformal field theory [$\mathcal{N} = 4$ super Yang-Mills $U(N)$ gauge theory] on its boundary $(S^3 \times \mathbb{R}) \times S^5$. Working at finite temperature (where the gauge theory is in a thermal state described by the Schwarzschild-AdS solution), one can show [2] that finite size effects on the gravity side become important at high temperatures $T \backsim 1/\ell$ (where ℓ is the AdS radius). The correspondence implies that the density of low-energy states is not affected even though the volume of S^3 has been reduced to S^3/Γ—hence, there must exist light states of the type described above.

At zero temperature, taking the quotient by Γ of AdS_5 produces an orbifold with fixed points at $r = 0$, and calculating the precise spectrum of twisted sector states is difficult due to the Ramond-Ramond background. However, there are suggestive arguments [2] that the AdS/CFT correspondence still predicts the existence of extra light states. The boundary energy of pure AdS_5 is exactly equal to the Casimir energy of $\mathcal{N} = 4$ super $U(N)$ Yang-Mills theory on S^3 with radius ℓ. Taking the quotient by Γ reduces both the energy and the volume by the same factor, leaving the energy density unchanged.

The preceding conclusions would be modified if solutions of Einstein's equations with a negative cosmological constant existed that asymptotically approached AdS_5/Γ but had less energy. If so, then the ground state energy density of the strongly coupled gauge theory on S^3/Γ would be even smaller than on S^3.

The existence of such solutions has been an open question until now. Here we show that there do exist solutions to Einstein's equations with a cosmological constant in five dimensions that are asymptotic to AdS_5/Z_p, where $p \geq 3$. They are obtained from a generalization of the Taub-Newman-Unti-Tamburino (Taub-NUT) metric [3] in a manner analogous to that used in deriving the AdS soliton. For a large cosmological constant, their spatial sections approach that of the Eguchi-Hanson (EH) metric [4] and so we call these solutions Eguchi-Hanson solitons. This is in contrast to the situation for the Kaluza-Klein monopole, in which it has been shown that there are no static monopole solutions to the five-dimensional Einstein equations with a cosmological constant that reduce to the asymptotically flat Kaluza-Klein monopole [5]. The total energy of an EH soliton is negative, though bounded from below, consistent with earlier arguments [2].

EH solitons are natural (five-dimensional) generalizations of the EH metric that can be derived from a set of inhomogeneous Einstein metrics on sphere bundles fibered over Einstein-Kahler spaces that were recently obtained [3,6]. Unlike the four-dimensional case, a Lorentzian signature is possible, thereby yielding a nonsimply connected background manifold for the CFT boundary theory.

To obtain the metric, we begin with the five-dimensional generalization [3] of the Taub-NUT metric,

$$ds^2 = -F(\rho)[d\tau + 2n\cos(\theta)d\phi]^2 + \frac{d\rho^2}{F(\rho)} + (\rho^2 + n^2)(d\theta^2 + \sin(\theta)^2 d\phi^2) + \rho^2 dz^2, \quad (1)$$

where the $U(1)$ fibration is a partial fibration over a two-dimensional subspace of the three-dimensional base space. The function $F(\rho)$ is

$$F(\rho) = \frac{4m\ell^2 - 2n^2\rho^2 - \rho^4}{\ell^2(\rho^2 + n^2)}, \quad (2)$$

with m a constant of integration. The condition $n^2 = \ell^2/4$ must hold for this to satisfy the 5D Einstein equations with positive cosmological constant $\Lambda = 6/\ell^2$. Because of this, the metric (1) does not have a sensible $\Lambda \to 0$ limit. In order to avoid singularities at $\theta = 0, \pi$, the coordinate τ must be identified with period $8\pi n$, yielding a spacetime with closed timelike curves.

We can obtain a solution with negative cosmological constant $\Lambda = -6/\ell^2$ through a judicious choice of analytic continuations $z \to it$, $\tau \to 2n\psi$, $\ell \to i\ell$, yielding the metric

$$ds^2 = -\rho^2 dt^2 + 4n^2\tilde{F}(\rho)[d\psi + \cos(\theta)d\phi]^2 + \frac{d\rho^2}{\tilde{F}(\rho)} + (\rho^2 - n^2)d\Omega_2^2, \quad (3)$$

$$\tilde{F}(\rho) = \frac{\rho^4 + 4m\ell^2 - 2n^2\rho^2}{\ell^2(\rho^2 - n^2)}, \quad (4)$$

where $d\Omega_2^2$ is the metric of the unit 2-sphere. By making the further transformations

$$\rho^2 = r^2 + n^2, \qquad m = \frac{\ell^2}{64} - \frac{a^4}{64\ell^2}, \quad (5)$$

and then setting $r \to r/2$, $t \to 2t/\ell$, we obtain

$$ds^2 = -g(r)dt^2 + \frac{r^2 f(r)}{4}[d\psi + \cos(\theta)d\phi]^2 + \frac{dr^2}{f(r)g(r)} + \frac{r^2}{4}d\Omega_2^2,$$
$$g(r) = 1 + \frac{r^2}{\ell^2}, \qquad f(r) = 1 - \frac{a^4}{r^4}, \quad (6)$$

which solves Einstein's equations with negative cosmological constant $\Lambda = -6/\ell^2$. Analytically continuing $\ell \to i\ell$ will turn (6) into a metric solving Einstein's equations with a positive cosmological constant.

The metric (6) provides us with a new means of obtaining the Eguchi-Hanson metric in four dimensions. In the $\ell \to \infty$ limit, the metric (6) yields the Eguchi-Hanson metric

$$ds^2 = \frac{r^2}{4}f(r)[d\psi + \cos(\theta)d\phi]^2 + \frac{dr^2}{f(r)} + \frac{r^2}{4}d\Omega_2^2 \quad (7)$$

as a $t =$ constant hypersurface. Note that the transformations (5) are crucial in obtaining this result; the metric (1) becomes degenerate in the $\ell \to \infty$ limit.

The metric (6) solves the Einstein equations with a negative (positive) cosmological constant $\Lambda = \mp 6/\ell^2$ (or the vaccum equations when $\ell \to \infty$). We call the metrics with $\Lambda < 0$ Eguchi-Hanson solitons, since they bear an interesting resemblance to the EH metric in four dimensions and have a solitonlike character similar to that of the AdS soliton [7]. However, unlike the four-dimensional case, a Lorentzian signature is possible, and, unlike the metric (1), there are no closed timelike curves (and no horizons when $\Lambda < 0$). Both the Ricci and the Kretschmann scalars are easily seen to be free of singularities.

However, stringlike singularities can arise at $r = a$ and must be dealt with separately. These can be eliminated in the usual way. Consider the behavior of the metric (6) as $r \to a$. Regularity in the (r, ψ) section implies that ψ has period $2\pi/\sqrt{g(a)}$, and elimination of string singularities at the north and south poles ($\theta = 0, \pi$) implies that ψ has period $\frac{4\pi}{p}$, where p is an integer. This implies in the asymptotically AdS case that

$$a^2 = \ell^2\left(\frac{p^2}{4} - 1\right), \quad (8)$$

with $p \geq 3$, yielding in turn that $a > \ell$.

We turn now to a computation of the energy of the EH soliton. Employing the boundary-counterterm method [8–12], we consider the general gravitational action and add to it the counterterm action [13], which depends only on quantities intrinsic to the boundary and, hence, leaves the equations of motion unchanged; it serves to cancel the divergences of the usual bulk/boundary actions. In five dimensions, it is

$$I = -\frac{1}{16\pi G}\left[\int_{\mathcal{M}} d^5x\sqrt{-g}(R - 2\Lambda) + 2\int_{\partial\mathcal{M}} d^4x\sqrt{\gamma}\Theta + \frac{2}{\ell}\int_{\partial\mathcal{M}} d^4x\sqrt{\gamma}\left(-3 - \frac{\ell^2}{2}\hat{R}\right)\right],$$

where γ is the induced metric on the boundary whose extrinsic/intrinsic curvature scalars are, respectively, Θ and $\hat{R}$. A conserved charge associated with a Killing vector ξ at infinity can be calculated using the relationship

$$\mathfrak{D}_\xi = \oint_\Sigma d^{d-1}S^a\xi^b T_{ab}^{\text{eff}}, \quad (9)$$

where $T_{ab}^{\rm eff}$ is obtained by varying the full action (including counterterms, up to the appropriate dimension—see Refs. [10,13] for its explicit form) with respect to the induced boundary metric, and $d^{d-1}S^a$ is the $(d-1)$-dimensional surface element density.

For asymptotically AdS spacetimes, $\mathfrak{D}_\xi$ is the conserved mass when ξ^a is a timelike-Killing vector; an analogous result holds the asymptotically dS case [12]. Thus, we find from the counterterm method that the conserved mass (or total energy) is

$$\mathfrak{M} = \frac{\pi(3\ell^4 - 4a^4)}{32G\ell^2 p} = \frac{(3\ell^4 - 4a^4)N^2}{16\ell^5 p}, \qquad (10)$$

where the second equality occurs because we can relate the parameters of the gravity theory in the bulk to those of the CFT on the boundary, $G = \pi\ell^3/(2N^2)$. It is straightforward to show that this is equal to the Euclidean action multiplied by the inverse of the (arbitrary) period of the Euclidean time parameter, yielding a solution with zero entropy, as expected for a horizonless metric.

We can compare the result (10) with that of the field theory on the boundary of the AdS_5 orbifold. Since the local geometry is unchanged (with only the volume of the S^3 becoming that of S^3/Γ), the calculation is the same as that for the AdS_5 case [14]. Rescaling the AdS orbifold metric by a factor of r^2/ℓ^2, as $r \to \infty$ we find the metric of the conformal field theory

$$ds^2 = -dt^2 + \frac{\ell^2}{4}[d\psi + \cos(\theta)d\phi]^2 + \frac{\ell^2}{4}d\Omega_2^2, \qquad (11)$$

which has a vanishing Weyl tensor. The stress tensor is, therefore, that of a field theory on a conformally flat spacetime in four dimensions whose expectation value $\langle \hat{T}^s_{ab}\rangle$ is a known quadratic function of the curvature with coefficients dependent upon the field content of the theory [15]. The energy is then given by

$$\mathcal{E} = \sum_{s=0,1/2,1} n^s \int_\Sigma d^3x\sqrt{\sigma}N_{lp}\langle \hat{T}^s_{ab}\rangle \xi^a u^b, \qquad (12)$$

where the sum is over the scalar, spinor, and vector fields of the field theory and where $\theta \in [0,\pi]$, $\phi \in [0, 2\pi]$, and $\psi \in [0, 4\pi/p]$. For $\mathcal{N} = 4$ super Yang-Mills theory, there are 6 scalars, 4 spinors, and 1 vector field [14]; inserting this information into (12), we obtain

$$\mathcal{E} = \frac{3N^2}{16\ell p}. \qquad (13)$$

Note that this matches the conserved mass given by (10) with $a = 0$. A straightforward computation using a Noether charge approach [16,17] of the energy of the soliton (6) relative to the AdS orbifold yields the difference $\mathfrak{M} - \mathcal{E}$ as expected. Note that, going p times along the ψ direction, the situation is the same as if the asymptotic space were S^3 (and not S^3/Γ), for which fermions are periodic. Hence, p must be even when fermions are present in the CFT.

We see that the energy (10) of the EH soliton is lower than that of the AdS_5/Γ orbifold and is, in fact, negative and finite once condition (8) is taken into account. Indeed, we have

$$\mathcal{E}_{\rm EH-soliton} = -\frac{(p^4 - 8p^2 + 4)N^2}{64\ell p} \qquad (14)$$

for any given integer $p \geq 3$. We conjecture that the EH soliton is the state of lowest-energy in its asymptotic class in both 5D Einstein gravity with a negative cosmological constant and in type IIB supergravity in ten dimensions. Indeed, the AdS/CFT correspondence (along with the expected stability of the gauge theory) suggests that any metric solving the 5D Einstein equations that has the same boundary conditions as the EH soliton will have a greater energy.

We now show that our conjecture holds perturbatively for all metrics of the form

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu},$$

where $\bar{g}_{\mu\nu}$ is the EH solition (6) and the perturbation obeys the falloff conditions

$$h_{\mu\nu} = \mathcal{O}(r^{-2}), \qquad h_{\mu r} = \mathcal{O}(r^{-4}),$$
$$h_{rr} = \mathcal{O}(r^{-6}), \qquad \mu, \nu \neq r.$$

Employing the method of Abbott and Deser [18], the Hamiltonian H on a time-symmetric slice to second order in the perturbation $h_{ij}(i, j \neq t)$ is

$$\mathcal{H} = \bar{N}\bigg[\frac{1}{\sqrt{\bar{g}}}p^{ij}p_{ij} + \sqrt{\bar{g}} \times\left(\frac{1}{4}(\bar{D}_k h_{ij})^2 + \frac{1}{2}{}^{(4)}\bar{R}^{ijkl}h_{il}h_{jk} - \frac{1}{2}{}^{(4)}\bar{R}^{ij}h_{ik}h_j{}^k\right)\bigg], \qquad (15)$$

where we have used the same notation, gauge, and constraint equations as in Ref. [7].

As the momenta make a positive contribution to the energy density, we need calculate only the gradient energy density $(\bar{D}_k h_{ij})^2$ (also positive) and the potential energy density $U = \frac{1}{2}{}^{(4)}\bar{R}^{ijkl}h_{il}h_{jk} - \frac{1}{2}{}^{(4)}\bar{R}^{ij}h_{ik}h^k_j$. We evaluate the latter by considering it as a matrix contracted with two 9-vectors whose components are h_{il}. We find that there exists a negative eigenvalue for sufficiently small r. Writing the perturbation as $h_{ik} = A(r)\tilde{h}_{ik}$, where $A(r)$ is a profile function maximized at $r = a$ and $\tilde{h}_{ik}$ is the eigenvector associated with the negative eigenvalue, we find that the negative potential energy U is not outweighed by a simple estimation of the gradient energy density (given by dividing the maximum of the profile function by the proper distance over which U is negative). This situation—quite unlike that for the AdS soliton [7]—forces us take into account the complete expression

$$T = \{\hat{h}_{ab}\partial_c A(r) + A(r)\hat{h}_{ab;c}\}^2 \tag{16}$$

for the gradient energy term. From this, we find that the gradient energy density always outweighs the potential energy density, indicating that the EH soliton is perturbatively stable for all values of p relative to all other metrics with the same boundary conditions.

We have also found $(d+1)$-dimensional generalizations of the EH solition (6). These are

$$ds^2 = -g(r)dt^2 + \left(\frac{2r}{d}\right)^2 f(r)\left[d\psi + \sum_{i=1}^{k}\cos(\theta_i)d\phi_i\right]^2 + \frac{dr^2}{g(r)f(r)} + \frac{r^2}{d}\sum_{i=1}^{k}(d\theta_i^2 + \sin^2(\theta_i)d\phi_i^2), \tag{17}$$

where

$$g(r) = 1 \mp \frac{r^2}{\ell^2}, \qquad f(r) = 1 - \left(\frac{a}{r}\right)^d, \tag{18}$$

and the cosmological constant $\Lambda = \pm d(d-1)/(2\ell^2)$. We shall discuss their derivation and properties more fully in a subsequent paper [19].

The EH soliton forms a new one-parameter family of solutions (indexed by p) to the 5D Einstein equations (and, hence, low-energy string theory) with a negative cosmological constant that asymptotically approach AdS_5/Γ. These solutions are perturbatively stable and of lower energy than AdS_5/Γ. It is natural to consider the extent to which the EH soliton has properties similar to that of the AdS soliton. The latter metric has been shown under certain conditions to be a unique lowest mass solution for all spacetimes in its asymptotic class [20]. It also satisfies holographic causality [21] and can undergo phase transitions to AdS black holes with Ricci-flat horizons [22]. Which of these properties are shared by the EH soliton remains an interesting subject for future work.

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