Collective Oscillation in a Hamiltonian System

Hidetoshi Morita[1,2,*] and Kunihiko Kaneko[2,3]
[1]Faculty of Science and Engineering, Waseda University, Shinjuku-ku, Tokyo 169-8555, Japan
[2]Department of Basic Science, The University of Tokyo, Komaba, Meguro-ku, Tokyo 153-8902, Japan
[3]ERATO Complex Systems Biology Project, JST, Meguro-ku, Tokyo 153-8902, Japan
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Oscillation of macroscopic variables is discovered in a metastable state of the Hamiltonian system of the mean-field *XY* model. The duration of the oscillation is divergent with the system size. This long-lasting periodic or quasiperiodic collective motion appears through Hopf bifurcation, which is a typical route in low-dimensional dissipative dynamical systems. The origin of the oscillation is explained, with a self-consistent analysis of the distribution function, as the self-organization of a self-excited swing state through the mean field. The universality of the phenomena is discussed.

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Dissipative systems often show periodic, quasiperiodic, and chaotic motion at a macroscopic level, when they are far away from equilibrium. The motion is described as low-dimensional dynamics, and its discovery has marked an epoch of nonlinear dynamics studies in physics. Recalling that the microscopic degrees of freedom involved are large, such macroscopic behavior is a result of collective motion that emerges out of high-dimensional microscopic dynamics. The collective motion, indeed, has been intensively and extensively studied for systems consisting of a large number of chaotic elements, as an important issue in high-dimensional dynamical systems [1–4]. The underlying microscopic dynamics in such studies, however, have been restricted to *dissipative* chaos.

Does such collective motion exist even in isolated thermodynamic systems, or Hamiltonian systems with many degrees of freedom? According to thermodynamics, any isolated macroscopic system finally relaxes to an equilibrium state, where any macroscopic variable remains constant with thermal fluctuation. Thus the only possibility for such macroscopic oscillation to exist in isolated systems lies in transient relaxation processes. In addition, for such collective motion to continue in a macroscopic time scale, long-term persistence of a nonequilibrium state is required. Indeed, long-term metastable states are found to exist, when some Hamiltonian systems are given certain initial conditions [5–7]. Yet, to our knowledge, no such collective low-dimensional motion has been reported in such metastable states.

Here we report the discovery of such collective oscillation of macroscopic (thermodynamic) variables in several Hamiltonian systems with many degrees of freedom. The oscillation is sustained over a long time, and indeed the duration increases with the system size, suggesting the divergence in the thermodynamic limit. Furthermore, the oscillation appears in a similar way to the bifurcation in low-dimensional dissipative dynamical systems, indicating the low dimensionality of the collective motion. In this Letter, the essence of this discovery is briefly reported, especially in the mean-field *XY* model [8].

We adopt the Hamiltonian system of the mean-field *XY* model, or globally coupled pendula [5,9,10],

$$\mathcal{H} = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} [1 - \cos(\theta_i - \theta_j)]. \quad (1)$$

All the N pendula interact with each other through phase differences. Each pendulum has two types of motion: rotation at a higher energy and libration at a lower energy. In the thermodynamic limit ($N \to \infty$), the equilibrium state is determined only with the total energy density $U = \mathcal{H}/N$, showing a continuous phase transition.

We focus on the dynamics of the variance of momentum, $T(t)$, and the modulus of the mean field, $M(t)$:

$$T(t) \stackrel{\text{def}}{=} \frac{1}{N} \sum_{j=1}^{N} p_j(t)^2, \qquad M(t) e^{i\phi(t)} \stackrel{\text{def}}{=} \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j(t)}. \quad (2)$$

These are macroscopic variables; indeed, they are nothing but the temperature and the magnetization of the system in equilibrium. Because (1) yields the constraint,

$$2U = T(t) + 1 - M(t)^2, \quad (3)$$

we mainly discuss the dynamics of $M(t)$ in the following. Using (2), the equations of motion are described as single pendula interacting with the mean field:

$$\dot{\theta}_j = p_j, \qquad \dot{p}_j = -M(t) \sin[\theta_j - \phi(t)]. \quad (4)$$

We give the initial conditions of $\{\theta_i\}$ and $\{p_i\}$, assigned with only two macroscopic parameters: the total energy density, U, and the initial magnetization, $M_0 = M(0)$. First, the initial distribution of $\{\theta_i\}$ is set as a Boltzmannian for the equilibrium state determined independently of U, with the magnetization M_0 and the temperature $T_{eq}(M_0)$, where $T_{eq}(M)$ is the equation of state in equilibrium:

$$f_\theta^0(\theta; M_0) = \frac{1}{Z_\theta(M_0)} \exp\left[\frac{M_0}{T_{\rm eq}(M_0)} \cos\theta\right], \quad (5)$$

where $Z_\theta(M_0)$ is the normalization. Next, the distribution of $\{p_j\}$ is set as a Maxwellian determined with the temperature $T_0(U, M_0) = 2U - 1 + M_0^2$ to fulfill (3):

$$f_p^0(p; U, M_0) = \frac{1}{Z_p(U, M_0)} \exp\left[-\frac{p^2}{2T_0(U, M_0)}\right], \quad (6)$$

where $Z_p(U, M_0)$ is the normalization. Here θ_i and p_i are given independently of each other. Then the initial distribution is $f_0(\theta, p; U, M_0) = f_\theta^0(\theta; M_0) f_p^0(p; U, M_0)$. In general, $T_{\rm eq} \neq T_0$, leading to initial states far from equilibrium, whereas if M_0 is the equilibrium value for the given U, then $T_{\rm eq} = T_0$, yielding exactly the equilibrium distribution. Thus these initial conditions are smoothly connected to equilibrium on the (U, M_0) plane.

Now we study relaxation phenomena far from equilibrium [11]. A typical time series of $M(t)$ is shown in Fig. 1. Initially, $M(t)$ decays almost exponentially. Then, however, $M(t)$ does not simply reach the equilibrium value but stays near a larger value for quite a long time. After this long interval, $M(t)$ departs from the plateau toward equilibrium. Finally, $M(t)$ reaches the equilibrium value and fluctuates about it. The duration of the plateau increases linearly with N (see the inset of Fig. 1), suggesting that the metastable state lasts in a macroscopic time scale in the thermodynamic limit.

Here we note that another metastable state in this model has been intensively investigated for a decade, especially by taking a rectangular (*water bag*) initial momentum distribution [5,6]. This metastable state exists only in the region just below the critical energy of the phase transition, and there $M(t)$ and $T(t)$ take smaller values than those in equilibrium, leading to a branch of negative specific heat. This state is regarded as a reflection of a stable stationary solution of the corresponding Vlasov equation. On the other hand, the metastable state that we have discovered takes larger values of $M(t)$ and $T(t)$ than those in equilibrium and exists over a much broader region than the negative specific heat branch. Thus the present metastable state is not explained by the above stationary solution of the Vlasov equation and is a novel one.

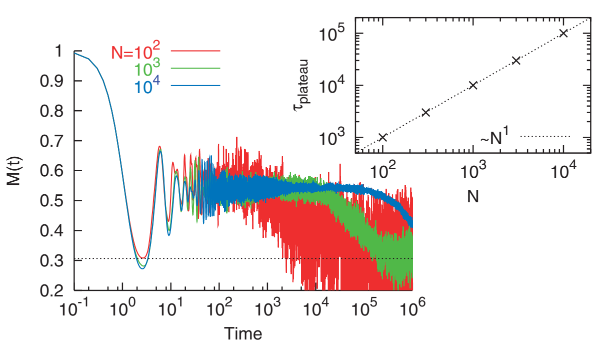

FIG. 1 (color). A time series of $M(t)$. The abscissa axis is a log scale. The dotted line is the equilibrium value. Inset: The duration of the plateau $\tau_{\rm plateau}$ against N. $U = 0.69$ [6] and $M_0 = 1$.

We now study the metastable state in more detail. The close-up time series of $M(t)$ in the metastable state shows a periodic oscillation [Fig. 2(a)]. The oscillation is not caused by the finiteness of N. Indeed, the oscillation becomes more apparent with increasing N, in strong contrast with fluctuation around equilibrium that reduces to zero. Its power spectrum [Fig. 3(a)] shows a large peak ($f \approx 0.166$), with its harmonic components, which remains sharp with increasing N. These indicate that the periodic motion survives even in the thermodynamic limit. In a longer run, the amplitude of the oscillation is not kept constant but decreases gradually (logarithmically in time). Its decay rate, however, reduces as $N^{-1/2}$ [Fig. 2(b)], which also confirms that the periodic motion lasts permanently in the thermodynamic limit.

As well as the periodic motion, $M(t)$ takes various temporal patterns depending on the parameter (U, M_0). In the region farther from equilibrium, the oscillation is not simply periodic; its power spectrum [Fig. 3(b)] indicates quasiperiodicity on a T^2 torus. In the region nearer to equilibrium, on the other hand, its behavior is almost stationary, the value being either the equilibrium value or some others.

The phase diagram of the collective motion on the (U, M_0) plane is depicted in Fig. 4. As shown, with increasing total energy, the temporal pattern of the macroscopic variable changes as stationary → periodic → quasiperiodic. This is regarded as a "bifurcation" of the collective motion. Here we note the similarity to the typical bifurcation route in low-dimensional dissipative dynamical systems, fixed point → limit cycle → torus, through Hopf bifurcations. Hence it is suggested that the present bifurcation of the collective motion is described as that of low-dimensional dynamical systems, in particular, by Hopf bifurcations.

We next investigate the bifurcation in more detail. The mean amplitude of $M(t)$ against U in the vicinity of the

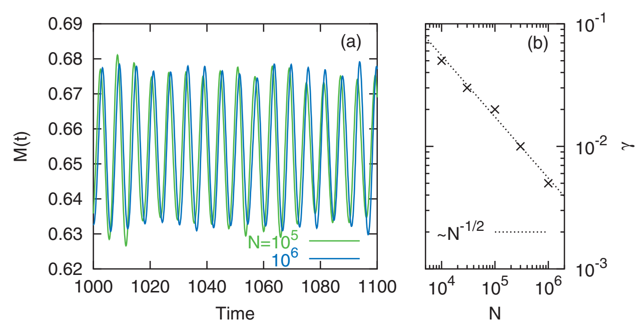

FIG. 2 (color). (a) A time series of $M(t)$ in the metastable state. (b) The decay rate γ of the amplitude of the oscillation, where $M_{\rm amp}(t) = M_{\rm amp}(t_0) - \gamma \log(t/t_0)$. $U = 0.5$ and $M_0 = 0.9$.

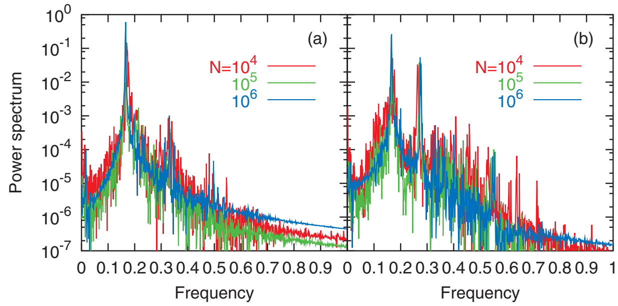

FIG. 3 (color). The power spectrum of $M(t)$ for (a) $U = 0.5$ [Fig. 2(a)] and (b) 0.75. $M_0 = 0.9$.

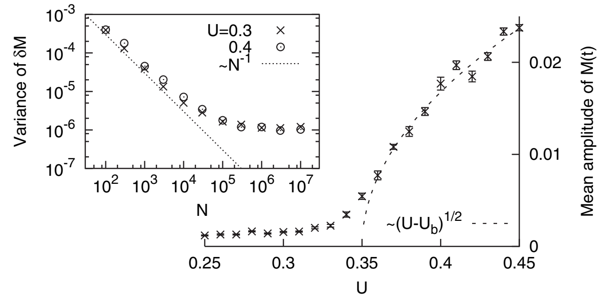

FIG. 5. The mean amplitude of $M(t)$ against U, in the vicinity of the bifurcation point $U_b = 0.35$. $N = 10^5$. The dashed line indicates $\sim(U - U_b)^{1/2}$. Inset: The system size dependence of the variance of the fluctuation δM around the mean stationary point ($U = 0.3$) or the Poincaré section of the mean periodic orbit ($U = 0.4$). Each datum is obtained as the average over 25 samples. $M_0 = 0.9$.

bifurcation point is shown in Fig. 5. The amplitude increases above some critical energy U_b, with the approximate dependence of $(U - U_b)^{1/2}$. This verifies that the bifurcation from the stationary to the periodic motion is a Hopf type. We have also studied the change of the amplitude against M_0, which is another parameter, and again confirmed the Hopf-type bifurcation from the stationary to the periodic motion [8].

Strictly speaking, however, there remains fluctuation of finite magnitude around each of the stationary, periodic, and quasiperiodic motions, even in the thermodynamic limit. The bifurcation is not completely one-dimensional but blurred. The fluctuation around the mean stationary value or around the periodic orbit remains finite with increasing N. In the inset of Fig. 5, the variance of $M(t)$ at $U < U_b$ and that around the Poincaré section of the mean periodic motion at $U > U_b$ are plotted for various system sizes. As shown, the variance of the fluctuations first decreases as $\sim N^{-1}$, but for larger N the decrease stops at a finite value, leading to a plateau. This behavior is quite similar to that studied in (dissipative) globally coupled maps [1]. This residual fluctuation is not represented by low-dimensional chaos, and may result from high-dimensional collective motion, as is observed in dissipative systems [1].

Next, we study the origin of the collective motion, especially the periodic motion, from the microscopic point of view. Because it is almost impossible to investigate the trajectories of such large degrees of freedom, we study the one-body distribution function, $f(\theta, p, t)$. A snapshot of $f(\theta, p, t)$ for the collective periodic motion when the phase of oscillation of $M(t)$ is zero (Fig. 6) shows a pair of high density regions at $((\theta - \phi)/2\pi, p) \approx (\mp 0.3, \pm 1)$, besides that at the center. The pair of peaks rotates clockwise along the separatrix keeping the localization, without diffusing out. Since this localized rotation actually results in macroscopic oscillation, it is important to ask its origin.

To answer the question, we consider an ensemble of N pendula independent of each other, parametrically driven with the periodic external force, $M_{\mathrm{ext}}(t) = g + h \sin\Omega t$:

$$\dot{\theta}_j = p_j, \qquad \dot{p}_j = -M_{\mathrm{ext}}(t) \sin\theta_j. \tag{7}$$

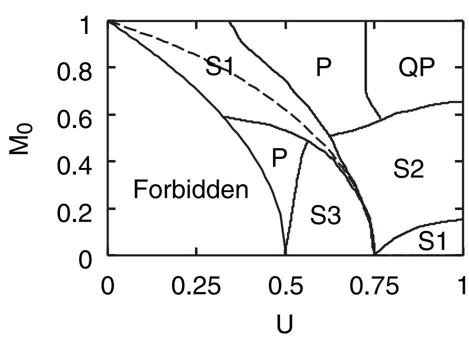

FIG. 4. The phase diagram. S1, S2, and S3 denote stationary states. In S1 the value of $M(t)$ is almost equal to the equilibrium value, in S2 it is larger, and in S3 it is smaller. P and QP denote the periodic and quasiperiodic motion. The dashed line indicates the equilibrium equation of state. The region $2U - 1 + M_0^2 < 0$ is forbidden, so that $T_0 \geq 0$. S3 may correspond to the negative specific heat branch [5,6].

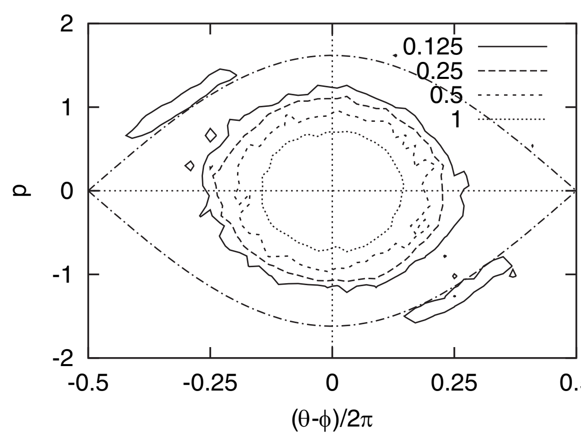

FIG. 6. A snapshot of the contour of the one-body distribution function, $f(\theta, p, t)$, at $t = 1001.2$ of Fig. 2(a). $N = 10^5$. The dash-dotted line indicates the separatrix.

These are almost equal to (4) except that $M_{\rm ext}(t)$ is externally applied, while $M(t)$ is self-consistently determined in (4). Indeed, let $M_{\rm int}(t)$ be the internal mean field given by (2), and if $M_{\rm int}(t) \equiv M_{\rm ext}(t)$, then the system is exactly the original Hamiltonian system. The Poincaré section of (7) on $\Omega t \equiv 0 \pmod{2\pi}$, an equivalent of the standard map [12], yields the islands of 1:1 resonance at $(\theta/2\pi, p) \approx (\mp 0.3, \pm 1)$, which are at just the same positions as the concentrated densities in Fig. 6. In addition, even though we first distribute the elements homogeneously, the axisymmetry is eventually broken down, while the point symmetry is preserved because of the requirement of the dynamics. For example, in a snapshot, the first and third (second and fourth) quadrants are dense (sparse).

On the basis of the above analysis, we give a self-consistent explanation on the collective periodic motion. Once a considerable number of elements gather in the islands of 1:1 resonance, they make a finite contribution to the periodicity of $M(t)$. The periodic $M(t)$ thus generated in turn drives the elements to stay in the island stably. The periodic drive also localizes other elements, breaking the axisymmetry, which can further stabilize the periodicity. Hence the collective periodic motion is the state of self-excitation through the mean field, or self-excited "swing," which is self-organized in the transient to equilibrium.

The above picture also accounts for the bifurcation from the stationary to the periodic state. An element requires some energy to stay in the island of 1:1 resonance. When the total energy is small, only a small number of elements can have such energy, which is too short to make the oscillation of $M(t)$ stable. When the total energy is large, on the other hand, many elements can have enough energy, which leads to stable periodic motion of $M(t)$. The bifurcation to periodic motion thereby appears with increasing U. The bifurcation along the M_0 axis follows the same scenario.

In summary, we have discovered the collective periodic and quasiperiodic motion in the metastable state of the Hamiltonian system of the mean-field XY model. In the thermodynamic limit, these oscillation states continue forever, as their lifetime seems to diverge. Similarly to the collective motion in dissipative systems, we have observed macroscopic low-dimensional motion arising from microscopic high-dimensional conservative chaotic dynamics. The collective oscillation appears through Hopf bifurcation, except for the residual fluctuation. The mechanism of the macroscopic periodicity is explained as the self-organization of the self-excited swing state.

The mechanism of the collective motion discussed above works in Hamiltonian systems if (i) the system has global, i.e., mean-field, coupling, and (ii) the resulting one-body dynamics has a separatrix of motion. We have also studied the mean-field ϕ^4 model, $H = \sum_i \{p_i^2/2 + (q_i^2 - 1)^2/4\} + (J/2N)\sum_{i,j}(q_i - q_j)^2/2$, which fulfills both of these conditions, and found the periodic oscillation of the macroscopic variable $Q = \sum_i q_i/N$, through the Hopf bifurcation [8], as studied here. Moreover, the first condition is relaxed to a simple long-range interaction. Indeed, we have confirmed the collective oscillation both in the XY and the ϕ^4 model on d-dimensional lattices with the coupling strength decreasing as $r^{-\alpha}$, r being the distance on the lattice, for $\alpha < d$ [8]. On the other hand, the existence of a separatrix in the one-body approximation implies a phase transition in terms of statistical physics. In nature, there are many systems with long-range interactions and phase transitions. Hence it will be an issue of interest to search for the collective motion that we have discussed in such real systems. Similar phenomena have been reported in a model of beam-plasma and free-electron laser systems [13]. Other possible examples may include molecular clusters and (bio)polymers.

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*Electronic address: morita@aoni.waseda.jp

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