Domain Instability during Magnetization Precession

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Spin wave equations describing the nonequilibrium precessional state of a ferromagnetic system are given. The equations reveal a new type of spin wave instability (SWI) towards growing domains of uniform magnetization. In the developed stages of SWI a nonstationary picture of domain chaos is revealed by numerical simulations. SWI is capable of explaining recent experimental observation of stochastic switching in precessional magnetization reversal.

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Fast and strong magnetic field pulses, generated by the highly relativistic electron bunches from a large electron accelerator, have opened the possibility to study precessional magnetization reversal [1–3]. In this fastest technique of magnetic switching, the magnetic field is spatially uniform on the lengths that are typical for a ferromagnetic system: the domain wall thickness, the grain size or the width of thin films, etc. The amplitude of the magnetic field in the pulse is such that the Zeeman energy exceeds the anisotropy energy. The integral of the magnetic field over the duration of the pulse can be varied to result in a uniform rotation of spins by angles ranging from few degrees to up to 2π [1]. Furthermore, the magnetic field vector may be applied at various angles with respect to the anisotropy axis [2]. During the pulse little excitation occurs to the ferromagnetic order of rotating spins. All these special properties of the fast pulses allow one to prepare in effect a nonequilibrium uniform ferromagnetic state with the magnetization vector pointing in a freely chosen arbitrary direction. This has led to precessional or ballistic switching, where traditional magnetic pulses of longer duration and weaker amplitude have been used. In this less transparent, but technically highly interesting case, the magnetization precession results from the combined action of the magnetic field during the pulse and the anisotropy [4–8].

After the pulse has vanished, precession of the magnetization starts in the anisotropy field. In this precession, which is almost periodic in time due to weak intrinsic dissipation, the magnetization vector may enclose large angles with the anisotropy axis. There is experimental evidence that such large angle precession is governed by the Landau Lifshits equation but with a Gilbert dissipation constant exceeding, by an order of magnitude, the one observed in ferromagnetic resonance where the precession angles are small [9]. In this Letter this large damping is related to the development of a spin wave instability (SWI) during the precession. SWI is similar to the two-magnon Suhl instability that has been extensively studied in connection with the parametric excitation of spin waves in a ferromagnet close to equilibrium [10,11]. SWI induces a fast transfer of magnetization from the uniform mode to the spin wave modes during small angle precession [12]. For the special case, in which the magnetization after the pulse is close to the saddle point of the anisotropy energy, SWI develops into growing domains of stable magnetization with the domain sizes being of the order of the domain wall thickness.

In this Letter, for ferromagnetic systems in the precessional state I find a system of spin wave equations in the rotating frame. Solution of these equations under general conditions yields SWI. The equation for the uniform mode coupled to the growing spin waves is the Landau Lifshits equations with the effective Gilbert constant that grows exponentially in time. Once SWI is developed, domains of saturated magnetization chaotic in space and time are revealed by numerical simulations.

In the precessional state the unit vector of local magnetization inside ferromagnetic particle is written as:

$$\vec{M}(t,\vec{r}) = \vec{n}(t)\sqrt{1-\vec{\phi}^2} + \vec{\phi}(t,\vec{r}), \tag{1}$$

$[\vec{M}^2(t,\vec{r}) = 1]$ using two-component spin wave field $\vec{\phi}(t,\vec{r}) = \phi^a(t,\vec{r})\vec{e}_a(t)$ $(a = 1, 2)$ orthogonal to $\vec{n}$ (spin waves are present either due to the thermal excitation before the pulse or due to SWI developing after the pulse) and by a rotating frame of three mutually orthogonal unit vectors $\vec{n}(t)$, $\vec{e}_a(t)$ at any time t. For $|\vec{\phi}| \ll 1$, the uniform precessional mode is defined as:

$$\vec{n}(t) = \frac{1}{ZV}\int \vec{M}(t,\vec{r})d^d\vec{r}, \tag{2}$$

where V and d are the volume and the dimension of a particle and normalization $Z \approx 1$. Two vectors $\vec{e}_a(t)$ are determined by the vector $\vec{n}(t)$ apart from the arbitrary time-dependent rotation around the axis $\vec{n}(t)$. This rotation represents $U(1)$ gauge invariance in the description of spin waves in a ferromagnet. Because $\vec{n}^2(t) = 1$, the vector $d\vec{n}/dt$ is orthogonal to $\vec{n}(t)$; therefore, it is convenient to choose $\vec{e}_1(t)$ to be proportional to $d\vec{n}/dt$. This gauge fixing condition is given by the Frenet-Serret equations: $d\vec{n}/dt = \vec{\zeta} \times \vec{n}$, $d\vec{e}_a/dt = \vec{\zeta} \times \vec{e}_a$, where the rotating frame is precessing as a solid structure about the Darboux vector $\vec{\zeta}(t) = -\tau(t)\vec{n}(t) + \kappa(t)\vec{e}_2(t)$, where $\kappa(t)$ is the local cur-

vature and $\tau(t)$ is the local torsion of the precessional curve $\vec{n}(t)$ on sphere $\vec{n}^2 = 1$.

At low temperatures only long wavelength spin waves are excited. Because the pulse induces a uniform coherent rotation short wavelength spin waves are not excited during the pulse and the use of the classical long wavelength dynamics is appropriate. Because of the Goldstone theorem the uniform mode $\vec{n}$ is absent in the exchange part of the Hamiltonian of a particle:

$$H_{\rm ex}[\phi] = \frac{J}{2}\int\left[(\partial_\mu\vec{\phi})^2 + \frac{(\vec{\phi}\cdot\partial_\mu\vec{\phi})^2}{1-\vec{\phi}^2}\right]d^d\vec{r}, \qquad (3)$$

where J is the exchange constant. Relativistic spin-orbit and dipole-dipole interactions give rise to anisotropy energy that for simplicity is written in the local approximation:

$$H_{\rm an}[\vec{M}] = \int E(\vec{M}(t,\vec{r}))d^d\vec{r}, \qquad (4)$$

where the anisotropy energy density $E(\vec{m})$ is an arbitrary function of $\vec{m}$ and does not depend on $\vec{r}$. Here and below $\vec{m}$ is an arbitrary unit vector distinct from the precessional mode $\vec{n}(t)$. One-ion anisotropy is an even function $E(\vec{m}) = E(-\vec{m})$ and is symmetric under the crystallographic group of transformations $\hat{T}_i$: $E(\hat{T}_i\vec{m}) = E(\vec{m})$. Uniform magnetic field can be included into $E(\vec{m})$. The choice of anisotropy as a uniform function of local magnetization Eq. (4) is very general and applies to many experiments. It holds for particles of special shapes like the ellipsoid where the dipole-dipole interaction reduces to an effective uniform demagnetizing field. At interfaces, where there is the Néel interface anisotropy energy due to the breakdown of the translational symmetry, Eq. (4) holds provided the spin wave modes with wave vectors perpendicular to the interface are not excited (as in ferromagnetic thin films).

The classical dynamics of local magnetization $\vec{M}(t,\vec{r})$ is described by the Larmor precession equation:

$$\partial_t\vec{M} = \gamma\vec{M}\times\frac{\delta H}{\delta\vec{M}}, \qquad (5)$$

where $\gamma = e/mc$ is the gyromagnetic ratio (we set $\gamma = 1$ by rescaling time $t \to \gamma t$) and $H = H_{\rm ex}[\vec{\phi}] + H_{\rm an}[\vec{M}]$ is the Hamiltonian. The Larmor equation conserves the total energy $dH/dt = 0$. We substitute into Eq. (5) $\delta H_{\rm ex}/\delta\vec{M} = \delta H_{\rm ex}/\delta\vec{\phi}$ and $\delta H_{\rm an}/\delta\vec{M} = -\vec{F}(\vec{M})$, where the force $\vec{F}(\vec{m}) = -dE/d\vec{m}$ is a vector field on a sphere $\vec{m}^2 = 1$. In the Frenet-Serret gauge: $\vec{e}_2\cdot\vec{F}(\vec{n}) = \kappa(t)$ and $\vec{e}_1\cdot\vec{F}(\vec{n}) = 0$. The total energy is the sum of the uniform mode anisotropy energy $E(\vec{n})$ and the spin wave energy $\mathcal{E}_{\rm sw}(\vec{n},[\vec{\phi}])$: $H = E(\vec{n}) + \mathcal{E}_{\rm sw}(\vec{n},[\vec{\phi}])$.

We project the Larmor equation Eq. (5) onto an equation representing the uniform mode Eq. (6) and onto an equation representing the remaining nonuniform modes Eq. (11). The component of both uniform and nonuniform equations that is parallel to $\vec{n}(t)$ is a linear combination of the other equations and therefore is redundant. For weakly excited spin waves we expand the Larmor equation up to the second power of the spin wave amplitudes. The equation for the uniform mode reads:

$$\frac{d\vec{n}}{dt} = \vec{n}\times\left[-(1+\Omega^0_{\rm sw})\vec{F}(\vec{n}) - \vec{n}\times\vec{F}(\vec{n})\frac{1}{\kappa^2}\frac{d}{dt}\mathcal{E}^0_{\rm sw}\right], \qquad (6)$$

where $\Omega^0_{\rm sw}(\vec{n},[\vec{\phi}])$ is the second order spin wave functional that gives the first correction to the speed of precession and $\mathcal{E}^0_{\rm sw}(\vec{n},[\vec{\phi}])$ is the second order spin wave energy:

$$\mathcal{E}^{\,0}_{\rm sw} = \frac{1}{2}\int[J(\partial_\mu\vec{\phi})^2 + \vec{\phi}\cdot\mathbf{C}\cdot\vec{\phi} + (\vec{n}\cdot\vec{F})\vec{\phi}^2]d^d\vec{r}, \qquad (7)$$

where $\mathbf{C}$ is the anisotropy mass tensor with components:

$$C_{\alpha\beta}(\vec{m}) = \frac{\partial^2 E}{\partial m^\alpha\partial m^\beta}, \qquad C_{ab}(\vec{n}) = e^\alpha_a\frac{\partial^2 E}{\partial n^\alpha\partial n^\beta}e^\beta_b, \qquad (8)$$

that can be related to the precession curvature $\kappa(t)$ and torsion $\tau(t)$. In the limit of small spin wave amplitudes the conservation of energy $E(\vec{n})$ in the Frenet-Serret frame gives: $\vec{n}\cdot\vec{F} + C_{11}(\vec{n}) = -\tau(t)$. Using $\kappa\vec{e}_1 = d\vec{n}/dt$ and $\kappa\vec{e}_2 = \vec{F} - \vec{n}(\vec{n}\cdot\vec{F})$ we find

$$C_{12}(\vec{n}) = \frac{1}{2\kappa(\vec{n})^2}\left(\frac{d\vec{n}}{dt}\cdot\frac{d}{d\vec{n}}\right)\kappa^2(\vec{n}) = \frac{d}{dt}\log\kappa(t). \qquad (9)$$

Equation (6) has the first integral: $E(\vec{n}) + \mathcal{E}^0_{\rm sw}(\vec{n},[\phi]) = \text{const}$, which is the total energy conservation.

Note that Eq. (6) has the form of the Landau Lifshits equation with the Gilbert dissipation constant

$$G(t) = \frac{1}{\kappa^2(t)}\frac{d}{dt}\mathcal{E}^0_{\rm sw}(\vec{n},[\vec{\phi}]). \qquad (10)$$

The sign of the Gilbert constant can be found in the vicinity of the extremum point of the anisotropy energy $E(\vec{m})$ provided the spin wave amplitudes grow. In the vicinity of the minimum point both the curvature $\kappa(t) > 0$ and the mass tensor $\mathbf{C}(\vec{n})$ approach zero and are proportional to the deviation angle from the minimum point. On the contrary the torsion $\tau(t)$ approaches some finite negative value. For fast growing spin wave amplitudes and negative $\tau(t)$ we find from Eq. (13): $d\mathcal{E}^0_{\rm sw}/dt > 0$. The Gilbert constant is positive and due to the total energy conservation $E(\vec{n}) + \mathcal{E}^0_{\rm sw} = \text{const}$: $dE(\vec{n})/dt < 0$. Therefore, the vector $\vec{n}(t)$ approaches the minimum point during the precession. Superficially this could be seen as the dissipation process although the nonuniform Larmor precession in ferromagnetic systems conserves the energy and is fully time reversible. The same analysis shows that near the maximum point the torsion $\tau(t)$ approaches some finite positive value and therefore the Gilbert constant is negative, $dE(\vec{n})/dt > 0$, and $\vec{n}(t)$ approaches the maximum point during the precession accompanied by SWI.

The equation for spin wave nonuniform modes reads:

$$\partial_t\vec{\phi} = \vec{n}\times[-J\partial^2_\mu\vec{\phi} + \mathbf{C}(\vec{n})\cdot\vec{\phi} + (\vec{n}\cdot\vec{F})\vec{\phi}], \qquad (11)$$

or in the rotating Frenet-Serret frame [13]:

$$\epsilon_{ab}\partial_t\phi^b = -J\partial_\mu^2\phi_a + C_{ab}(\vec{n})\phi^b + (\tau + \vec{n}\cdot\vec{F})\phi^a. \tag{12}$$

This linear system is Hamiltonian with the first integral

$$\mathcal{E}^0_{\rm sw}(\vec{n}, [\vec{\phi}]) + \tau(t)\int[\vec{\phi}(t,\vec{r})]^2 d^d\vec{r} = \text{const}. \tag{13}$$

In the limit of infinitely small amplitudes of all spin waves the anisotropy energy $E(\vec{n})$ is approximately conserved and the uniform mode precesses periodically in time $\vec{n}(t) = \vec{n}(t+\mathcal{T})$, where $\mathcal{T}$ is the period of precession. Solution of Eq. (12) for a spin wave mode with momentum $\vec{q}$ can be written as a monodromy matrix:

$$\vec{\phi}(t+\mathcal{T}) = \mathbf{M}(t,\mathcal{T},\vec{q})\vec{\phi}(t), \tag{14}$$

with unit determinant $\det\mathbf{M} = 1$. Let $a(\mathcal{T}, q) = \mathrm{Tr}\,\mathbf{M}/2$. If $|a| < 1$ then the eigenvalues of $\mathbf{M}$ are two complex numbers $\lambda_{1,2} = a \pm i\sqrt{1-a^2}$ otherwise ($|a| \geq 1$) they are real: $\lambda_{1,2} = a \pm \sqrt{a^2-1}$. In the latter case spin wave mode is instable and its amplitude grows exponentially with the increment $\nu(\vec{q}) = \log|\lambda_1|$. Two examples have been studied numerically: $E(\vec{m}) = \lambda_x m_x^2 + \lambda_y m_y^2 + \lambda_z m_z^2$ and $E(\vec{m}) = m_x^4 + m_y^4 + m_z^4$. In the first case the uniform precession: $\vec{n}(t), \vec{e}_1(t), \vec{e}_2(t)$ is found analytically in terms of the elliptic functions. Numerical solution in both cases shows that $a(\lambda)$ is an oscillating (but not periodic) function of spectral parameter $\lambda = Jq^2$. Note, that for $C_{ab} = 0$, $a(\lambda) = \cos(\lambda\mathcal{T})$. There are infinitely many intervals of instability [zones $|a(\lambda)| \geq 1$]; in both cases, they appear periodically and for $\lambda \to \infty$ they shrink. The spin wave dynamics in the periodic precessional state of a ferromagnet is similar in many ways to the dynamics of particle in 1D periodic potential. In the latter case it is well known that gaps in the spectrum open that separate the conduction bands. These gaps correspond to the instability zones in our case. We propose without proof that for general precession there exist unstable spin wave modes in the momentum space.

In the saddle point where $\vec{n}$ is constant and for special precessions the tensor $\mathbf{C}$ does not depend on time. In this case the spin wave dispersion in Eq. (12) reads:

$$\omega(\vec{q}) = \sqrt{Jq^2(Jq^2 + C_{22} - C_{11}) - C_{12}^2}. \tag{15}$$

SWI develops if either $C_{12} \neq 0$ or $C_{22} - C_{11} < 0$. Therefore, systems with an easy-axis anisotropy are always unstable whereas easy-plane systems could be stable provided $C_{12} = 0$, yet this requires precession along a circle with the constant curvature Eq. (9). An example of SWI is shown in Fig. 1 for 1D exchange coupled easy-axis chain of 512 spins. It is described in the long range limit by the Hamiltonian (3) and (4), where $E(\vec{m}) = -\lambda m_z^2$, $J = 1$, and $\lambda = 0.009$. The initial magnetization is set to $M^z(0,x) = 0.5 + 0.01\sin(3\pi x/128)$, $M^y(0,x) = 0$, and $M^x = \sqrt{1-M_z^2}$. x is an integer coordinate of sites. The Larmor equation (5) is solved with the time measured in units $1/J$. Four magnetization profiles $M^z(t,x)$ are shown for $t = 0$, 400, 600, 800 with increasing amplitudes and higher harmonics. Components M^x and M^y perform uniform precession accompanied by SWI (not shown). We observe that $M^z(t,x)$ remains periodic in space x but it seems to be periodic in time, too, with period $\mathcal{T}_{\rm sw} \approx 1600$. At times $t > 800$ (not shown) the amplitude decreases and reaches the initial condition profile at $\mathcal{T}_{\rm sw}$ [14].

The dispersion (15) has a branch with positive imaginary part: $\omega_A(q) = i\nu(q)$ in the region $0 < q < q_i$, where $2Jq_i^2 = \sqrt{(C_{22}-C_{11})^2 + 4C_{12}^2} - (C_{22} - C_{11})$. The spin wave distribution function in momentum space: $N_{\vec{q}}(t) = \vec{\phi}^2(t,\vec{q})$ grows exponentially with the increment reaching its maximum $\nu_m = \nu(q_m)$ at $q = q_m$, where $2Jq_m^2 = C_{11} - C_{22}$ or $q_m = 0$. At $t \to \infty$ the spin wave energy Eq. (13) is the sum over all instable modes:

$$\mathcal{E}^0_{\rm sw}(\vec{n}, [\vec{\phi}]) = -V\tau(t)\int_{q<q_i} N_{\vec{q}}(t)\frac{d^d\vec{q}}{(2\pi)^d}. \tag{16}$$

Growing instability disturbs the spin wave distribution in the momentum space away from the thermal equilibrium distribution $N^0_{\vec{q}} = T/\omega_0(\vec{q})$, where $\omega_0(\vec{q})$ is the spin wave dispersion before the pulse. After the pulse, SWI modes grow and the scattering in the direction of the equilibrium distribution grows, too. Both processes can be described by the Boltzmann kinetic equation:

$$\partial_t N_{\vec{q}} = 2\nu(q)N_{\vec{q}} - I(N_{\vec{q}}), \tag{17}$$

where the relaxation rate is given by the collision integral $I(N_{\vec{q}})$. Generally $\nu(q)$ and $I(N_{\vec{q}})$ are of the same order of magnitude. The leading contribution to the collision integral comes from the two-magnon scattering with one magnon having large momentum $\vec{p}$ in the incoming and outgoing states and second long wavelength magnon with the momentum $\vec{q}$ in the instability region $0 < q < q_i$:

$$I(N_{\vec{q}}) = \int W(\vec{q},\vec{p})(N_{\vec{q}} - N_{\vec{p}})\frac{d^d\vec{p}}{(2\pi)^d}, \tag{18}$$

where the scattering probability does not depend on momentum: $W(\vec{q},\vec{p}) = W$, and is small at low temperatures $W \sim (T/J)^{5/2}$. It could be proven that the probability of absorption and emission of long wavelength magnon with purely imaginary dispersion $\omega_A(q) = i\nu(q)$ by a short wavelength magnon is zero. The probability of two long wavelength magnon scattering is proportional to the small factor $[\nu(q)/J]^2$ and could be neglected.

Solution of the Boltzmann kinetic equation (17) yields:

$$N_{\vec{q}}(t) = [N^0_{\vec{q}} + \Phi(t)]\exp[(2\nu(q) - \chi)t], \tag{19}$$

where $\Phi(t) = N_0[\exp(\chi t) - 1]$, $\chi = W\,\Omega$ is the relaxation rate, $\Omega = \int_{q<q_i} d^d\vec{q}/(2\pi)^d$ is the volume of insta-

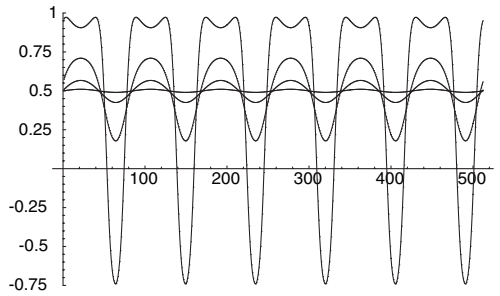

FIG. 1. Four magnetization profiles $M^z(x)$ during growth of SWI as it is explained in the text.

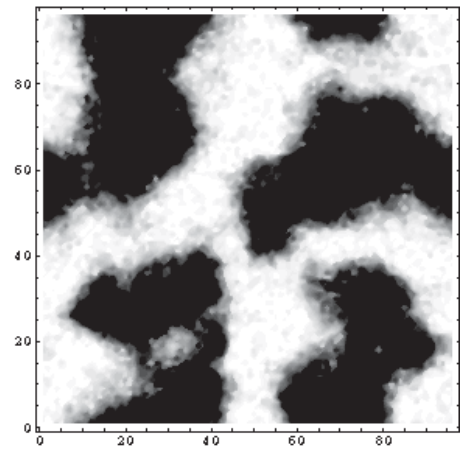

FIG. 2. Chaotic domains in the cluster of 96×96 exchange coupled classical spins with easy-axis anisotropy $E(\vec{m}) = -\lambda m_z^2$, $\lambda = 0.02J$. Initial conditions are random: $M^x(0, \vec{r}) \approx 1$, $|M^{y,z}(0, \vec{r})| \ll 1$. Nonlinearity $\vec{M}^2 = 1$ cuts the growing spin waves into multidomains. Shades of gray represent $M^z(\vec{r})$ at $t = 290/J$, with black and white being spin-up and -down.

bility region in the momentum space, and $N_0 = \int_{q<q_i} N_{\vec{q}}^0 d^d\vec{q}/(2\pi)^d\ \Omega$ is the average number of would-be instable spin waves in the system per one mode before the pulse arrives. Solution of Eq. (19) has an asymptote: $N_{\vec{q}}(t) = N_0 \exp[2\nu(q)t]$, as $t \to \infty$ that does not depend on the initial occupation number for this mode. The spin wave energy Eq. (16) at $t \to \infty$ could be estimated by the steepest descent method.

In particles with the size less then the critical size L_c, SWI could not develop due to the wave vector quantization. L_c depends on the shape of the particle and is proportional to the domain wall width $L_c \sim \sqrt{J/\lambda}$.

For vanishing dissipation ($G_0|\dot{\vec{n}}| \ll \nu_m$) the magnetization dynamics Eq. (5) could be traced during times much larger then the instability time ν_m^{-1}. At long times spin waves saturate (as in Fig. 1) to a domain pattern that depends on the initial condition for the magnetization. If initially the magnetization is close to the saddle point of the anisotropy energy with small thermally exited spin waves, then a special multidomain state emerges at $t\nu_m \gg 1$ (see, for example, Fig. 2) and it evolves chaotically in time and space. In this example the total magnetization is conserved $n^z(t) = 0$; therefore, the area of black and white domains are identical. The local width of domain walls in Fig. 2 fluctuates and is not determined by the minimum of energy as in equilibrium.

In the absence of SWI the magnetization of the easy-axis ferromagnetic particle, excited to the vicinity of the saddle circle by the pulse, spirals down deterministically into the nearest minimum of the anisotropy energy. But in large particles SWI is developing into the domain chaos and new possibility arises. In granular systems, weak random anisotropy at boundaries or the dipolar interaction destroys the conservation of $n^z(t)$. This might induce a diffusion of magnetization $n^z(t)$ due to the microscopic random torques applied at the chaotic domain wall positions. The total energy of multidomains will monotonically relax due to the weak dissipation and eventually a single domain state prevails. If the dissipation allows enough time for $n^z(t)$ to diffuse into the adjacent energy well (change the sign) then the fate of this particle becomes probabilistic with both minima of the anisotropy energy being possible, depending on the thermal initial conditions. This then manifests itself in the stochasticity of precessional switching observed in Ref. [3].

In conclusion, we have found the microscopic equations that govern a new type of general spin wave instability in the precessional state of a ferromagnetic system.

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[13] If the dipolar interaction is important then the anisotropy mass tensor is modified (the spin wave dispersion term): $\delta C_{\rm ab}(t, \vec{q}) \sim [\vec{q} \cdot \vec{e}_a(t)][\vec{q} \cdot \vec{e}_b(t)]/|q|^{d-1}$.
[14] Similar to the auto-oscillations in the spin wave turbulence, see, e.g., [11].