Saturated Ferromagnetism from Statistical Transmutation in Two Dimensions

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The total spin of the ground state is calculated in the $U \to \infty$ Hubbard model with uniform magnetic flux perpendicular to a square lattice, in the absence of Zeeman coupling. It is found that the saturated ferromagnetism emerges in a rather wide region in the space of the flux density ϕ and the electron density n_e. In particular, the saturated ferromagnetism at $\phi = n_e$ is induced by the formation of a spin-$1/2$ boson, which is a composite of an electron and the unit flux quantum.

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Ferromagnetism remains a challenging problem in spite of being among the best known phenomena in condensed matter physics. In particular, the origin of ferromagnetism in electron systems is fundamentally quantum mechanical and nonperturbative [1]. There are rather few established mechanisms of ferromagnetism, especially those of the saturated (complete) ferromagnetism. Saturated ferromagnetism is defined as the ground state of the many-electron system *with spin-independent interaction* having the maximum possible total spin.

Nagaoka's theorem is one of few rigorous results on the saturated ferromagnetism [2]. It guarantees that the saturated-ferromagnetic state is the unique ground state when a single hole is inserted in the half-filled Hubbard model with infinite on-site repulsion U. Unfortunately, this theorem is limited to the single-hole case. Numerical studies suggest that Nagaoka ferromagnetism is unstable in the thermodynamic limit at finite hole densities [3].

The flat-band ferromagnetism is another rigorous result [4,5]. Namely, the saturated ferromagnetism is proved rigorously under certain conditions, in electron systems with a (nearly) flat dispersion in the lowest band for a single electron. The ferromagnetism in a system with the low electron density and singular density of states near the Fermi level [6] may also be related to the flat-band mechanism. However, as these results still have limited applicability, it is worth pursuing other mechanisms of (saturated) ferromagnetism.

The difficulty in realizing the ferromagnetism in many-electron systems may be attributed to the Pauli principle for electrons. In the absence of interaction, the ground state of the system is generally paramagnetic, because the lower energy bands are filled with up and down spins. If we consider a system of bosons rather than fermions, the intrinsic tendency to favor paramagnetism may be absent. In fact, it was proved that, in a continuous system with spinful bosons, one of the ground states is always fully polarized if explicit spin-dependent interactions are absent [7,8]. This statement holds also in a lattice model [8,9]. In particular, for the infinite-U Hubbard model with spin-$1/2$ bosons, the total spin of the ground state is shown to be maximal *for all hole densities*, unlike in the electronic Hubbard model [9].

Thus, the saturated ferromagnetism could emerge in an electron system if the statistics of the electron is transmuted to bosonic. In fact, the statistical transmutation is indeed possible in two-dimensional (2D) systems [10,11], and it has been applied to fractional quantum Hall effect [12].

Combining these ideas, we can expect that the 2D electron system in the presence of an external gauge (magnetic) field exhibits the (saturated) ferromagnetism thanks to the formation of the composite boson. Namely, when the applied magnetic field amounts to unit flux quantum per electron, the magnetic-flux quantum may be assigned to an electron. The composite particle consisting of an electron and the attached flux is then expected to have spin $1/2$ and to obey the Bose statistics with hard-core constraint. In this way, in the mean-field level, the original system can be mapped into a spin-$1/2$ boson system without magnetic field, which exhibits the saturated ferromagnetism [7–9]. However, as the "flux attachment" argument is not rigorous, whether this mechanism actually leads to the ferromagnetism in an electron system has to be checked.

In quantum Hall systems in the continuum, the fully spin-polarized ground state is favored for the filling factor $\nu = 1$ (and, in general, for $\nu = 1/m$ with m odd), without the Zeeman energy [13–15]. This is referred to as quantum Hall ferromagnets [16]. While this ferromagnetism is usually associated with the antisymmetric nature of the orbital part of the wave function, it could also be regarded as a consequence of the formation of a spin-$1/2$ boson which is composed of an electron and m flux quanta [16]. On the other hand, the dispersion in quantum Hall systems in the continuous space is completely flat (Landau levels). Thus the saturated ferromagnetism may also be understood as a special case of the flat-band ferromagnetism [17]. It is not clear whether the statistical transmutation is essential to realize the saturated ferromagnetism in quantum Hall systems.

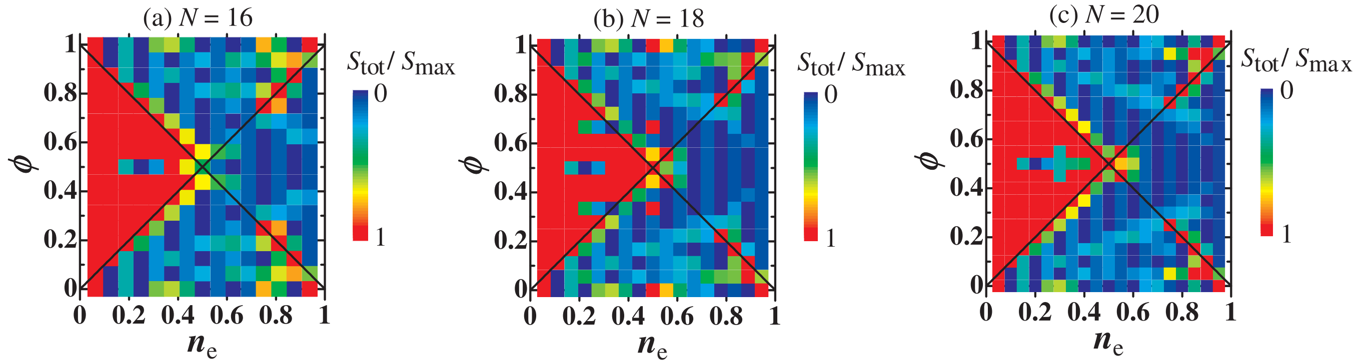

FIG. 1 (color). Total spin of the ground state as functions of the electron density (n_e) and the flux per plaquette (ϕ) in the 2D $U \to \infty$ Hubbard model. Solid lines are straight lines with $\phi = n_e$ and $\phi = 1 - n_e$, where statistical transmutation is expected.

In order to clarify this issue, in this Letter we study an electron system on a lattice. We demonstrate an example of saturated ferromagnetism which is due entirely to statistical transmutation and is distinct from the flat-band variety.

Let us introduce the $U \to \infty$ Hubbard model on a square lattice, with the gauge (magnetic) flux ϕ per plaquette [18]:

$$\mathcal{H} = -\sum_{\langle ij\rangle\sigma}[t_{ij}(\phi_{ij})c^\dagger_{i\sigma}c_{j\sigma} + \text{H.c.}] + U\sum_i n_{i\uparrow}n_{i\downarrow},$$
$$t_{ij}(\phi_{ij}) \equiv t\exp(i2\pi\phi_{ij}/\phi_0), \tag{1}$$
$$\phi = \sum_{\text{oriented plaquette}}\phi_{ij}, \qquad \phi_0 \equiv h/e \equiv 1,$$

where $\langle ij\rangle$ refers to the nearest-neighbor pairs. Periodic boundary conditions are imposed in both directions, unless explicitly mentioned otherwise. In the $U \to \infty$ limit, we have

$$\mathcal{H} = -\sum_{\langle ij\rangle\sigma}[t_{ij}(\phi_{ij})\tilde{c}^\dagger_{i\sigma}\tilde{c}_{j\sigma} + \text{H.c.}], \tag{2}$$

where $\tilde{c}_{i\sigma} = c_{i\sigma}(1 - n_{i,-\sigma})$, which means that double occupancy at each site is excluded. We stress that in our model (2) we have not included the Zeeman coupling of spins to the magnetic field. The model is therefore completely isotropic in the spin space, and the total spin is a conserved quantum number.

Exact numerical diagonalization for 4×4, $\sqrt{18} \times \sqrt{18}$, and $\sqrt{20} \times \sqrt{20}$ clusters is employed in our study. We study the system with various values of the electron density n_e and the flux per plaquette ϕ. In particular, we need to investigate the case $\phi = n_e$ where the statistical transmutation to boson would occur. Under periodic boundary conditions, the total flux of the system is quantized to an integer. Thus ϕ can take only integral multiples of $1/N$, where N is the number of sites (plaquettes). In order to study all the possible values of ϕ, we need to use the string gauge [19]: Choosing a plaquette S as a starting one, we draw $N - 1$ outgoing arrows (strings) from the plaquette S, so that each plaquette other than S is the endpoint of a string. Then we set ϕ_{ij} on a link ij to $\phi\mathcal{N}_{ij}$ taking account of the orientation, where $\mathcal{N}_{ij}$ is the number of strings cutting the link ij. We have checked that the single-electron spectrum for a small system size ($N \sim 20$) in the string gauge approximately reproduces the Hofstadter butterfly in the thermodynamic limit [20].

The total spin S_{tot} at zero temperature can be evaluated from the expectation value of $(\boldsymbol{S}_{tot})^2 = (\sum_\ell \boldsymbol{S}_\ell)^2$ in the ground state, where $\boldsymbol{S}_\ell$ is the spin operator at site ℓ. In Fig. 1 we show the scaled total spin S_{tot}/S_{max} in the ϕ-n_e plane. Here S_{max} is given by $N_e/2$ with N_e being the number of electrons. Red regions correspond to saturated-ferromagnetic states. In addition to Nagaoka ferromagnetism in the single-hole case with $\phi = 0$, we find two common features irrespective of the system size: (i) Saturated ferromagnetism appears along a straight line with $\phi = n_e$ (or $\phi = 1 - n_e$) except $0.6 \lesssim n_e \lesssim 0.7$.

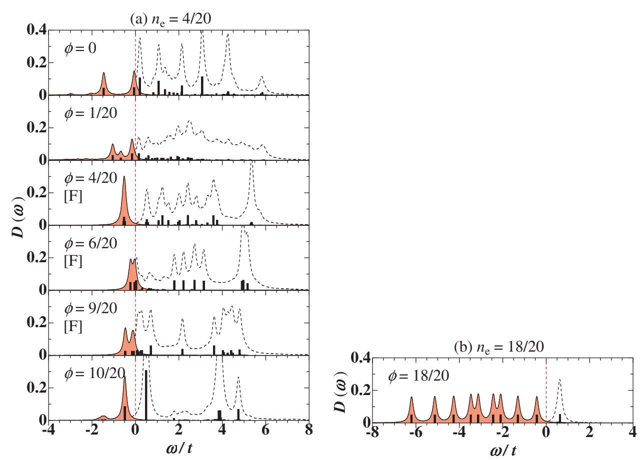

FIG. 2 (color). Spectral functions $D^\pm$ in the 2D $U \to \infty$ Hubbard model. (a) $n_e = 4/20$ with various values of ϕ; (b) $n_e = 18/20$ with $\phi = 18/20$. The delta functions (vertical bars) are broadened by a Lorentzian with a width of $0.1t$. The chemical potential is located at the zero energy. Cases with [F] exhibit saturated ferromagnetism. Colored regions represent D^-, while the dashed curves show D^+.

(ii) Saturated ferromagnetism appears in the triangular region surrounded by three straight lines: $n_e = 0$, $\phi = n_e$, and $\phi = 1 - n_e$.

Result (i) confirms the expectation based on the statistical transmutation. Moreover, we find that, in most cases of $\phi = n_e$, the saturated ferromagnetism is robust against twisting the boundary condition. This is in contrast to Nagaoka ferromagnetism, where the total spin of the ground state is changed from maximum to zero as the boundary is twisted [21]. Besides, we have checked for $N = 16$ and 18 that the saturated-ferromagnetic ground state at $\phi = n_e$ is nondegenerate except for the trivial $(2S_{\max} + 1)$-fold degeneracy. This is consistent with the ferromagnetism for spin-1/2 bosons [8,9].

In order to distinguish the ferromagnetism due to the statistical transmutation from possible "flat-band" varieties, we define the spectral functions

$$D^-(\omega) = \frac{1}{N}\sum_{\ell,n} |\langle \Psi_n(N_\uparrow - 1, N_\downarrow; \phi)| c_{\ell\uparrow} |\Psi_0(N_\uparrow, N_\downarrow; \phi)\rangle|^2 \times \delta(\omega + E_n(N_\uparrow - 1, N_\downarrow; \phi) - E_0(N_\uparrow, N_\downarrow; \phi) + \mu), \tag{3}$$

$$D^+(\omega) = \frac{1}{N}\sum_{\ell,n} |\langle \Psi_n(N_\uparrow + 1, N_\downarrow; \phi)| c^\dagger_{\ell\uparrow} |\Psi_0(N_\uparrow, N_\downarrow; \phi)\rangle|^2 \times \delta(\omega - E_n(N_\uparrow + 1, N_\downarrow; \phi) + E_0(N_\uparrow, N_\downarrow; \phi) + \mu). \tag{4}$$

Here μ is the chemical potential and $|\Psi_n(N_\uparrow, N_\downarrow; \phi)\rangle$ denotes an eigenstate with energy $E_n(N_\uparrow, N_\downarrow; \phi)$ in the system with $N_\uparrow$ up spins, $N_\downarrow$ down spins, and the flux ϕ. We define the index n so that $n = 0$ corresponds to the ground state with the given $N_\uparrow$ and $N_\downarrow$. In the following, we set $N_\uparrow = N_\downarrow = N_e/2$ for N_e even. $D^\pm$ can be estimated numerically by the continued-fraction method [22]. Below, we show the results for two values of n_e; $n_e = 4/20$ and $n_e = 18/20$ as representatives of the "low electron density" and "high electron density" regimes, respectively.

First let us focus on the low electron density case, $n_e = 4/20$. Figure 2(a) shows the evolution of $D^\pm(\omega)$ with varying ϕ. The sum of D^+ and D^- corresponds to the density of states (either occupied or unoccupied). Apparently it is always spread over a similar range of energy, representing the "bandwidth" which is about $8t$ although there is some ϕ dependence.

On the other hand, at this density, we find a crucial difference in the spectral function D^- corresponding to magnetism. Namely, D^- is concentrated in a narrow range of energy when the system exhibits the saturated ferromagnetism for $\phi = 4/20, 5/20, \cdots, 9/20$ (and $1 - \phi$). In contrast, when the saturated ferromagnetism is absent, D^- is spread over a region of energy. Intuitively, the spectral function D^- corresponds to the density of states *occupied by electrons*. The narrow distribution of D^- compared to the bandwidth indicates a variant of the narrow or nearly flat-band ferromagnetism [6].

In fact, there is more difference in D^- between the cases with and without saturated ferromagnetism than what is visible in Fig. 2. D^- vanishes completely below a certain threshold ($\omega/t = -0.550$, -0.229, and -0.474 for $\phi = 4/20$, $6/20$, and $9/20$, respectively) when the system exhibits the saturated ferromagnetism. In contrast, there is a continuous "shoulder" of low intensity (invisible in Fig. 2) down to much lower energy $\omega/t \sim -20$ when saturated ferromagnetism is absent. This seems to be consistent again with our interpretation.

In particular, for $\phi = 4/20 (= n_e)$, the spectral function D^- is localized within a narrow energy band below an apparent gap around the Fermi level. This appears similar to the quantum Hall ferromagnet at $\nu = 1$ in the continuum [16]. In this case, the statistical transmutation mechanism and the flat-band mechanism appear indistinguishable.

Now let us discuss the saturated ferromagnetism observed in the high electron density regime, at $\phi = n_e = 18/20$. Figure 2(b) shows the spectral functions in this case. Clearly, the spectral function D^- spreads over almost the entire bandwidth even though the system does exhibit the saturated ferromagnetism. Thus the ferromagnetism at $\phi = n_e = 18/20$ is difficult to be understood in terms of the flat-band mechanism, and appears to be exclusively due to the statistical transmutation mechanism. In fact, changing the value of ϕ destroys the saturated ferromagnetism at this electron density, as expected from the statistical transmutation scenario.

In order to further confirm the statistical transmutation scenario at $\phi = n_e$, we define the following operators: $b_{\ell\sigma} = e^{-i\mathcal{J}_\ell} c_{\ell\sigma}$ and $b^\dagger_{\ell\sigma} = c^\dagger_{\ell\sigma} e^{i\mathcal{J}_\ell}$, where $\mathcal{J}_\ell = -m\sum_{i(\neq\ell)} \theta_{\ell i} n_i$ with $n_i = \sum_{\sigma=\uparrow,\downarrow} c^\dagger_{i\sigma} c_{i\sigma}$. In general, m denotes the number of magnetic-flux quanta, and we set $m = 1$ in the present case. $\theta_{\ell i}$ is the argument of the vector drawn from site i to site ℓ. Note that the relation $\theta_{\ell i} -$

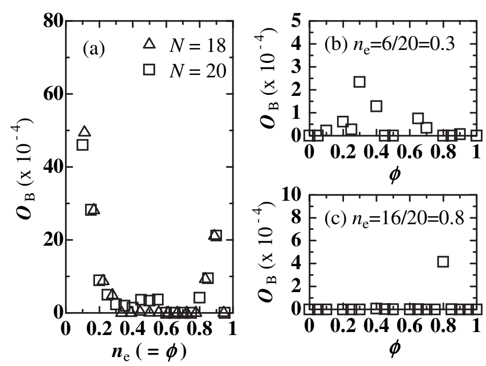

FIG. 3. Order parameter of spin-1/2 bosons in the 2D $U \to \infty$ Hubbard model. (a) $n_e (= \phi)$ dependence; (b) ϕ dependence for $n_e = 6/20$; (c) ϕ dependence for $n_e = 16/20$. In (b) and (c), data are not plotted where the initial and/or final states in Eq. (5) have degeneracy.

$\theta_{i\ell} = \pm\pi$ holds for $i \neq \ell$. Then we can prove that these operators satisfy the commutation relations of bosons for two different sites. There is a large freedom in determining explicit values of $\theta_{i\ell}$. Here we follow a prescription introduced in Ref. [23], although this prescription unavoidably breaks translation invariance of a periodic cluster. The order parameter for the condensation of the composite bosons may be defined by

$$O_{\rm B} = \frac{1}{N}\sum_{\ell}|\langle\Psi_0\Big(N_\uparrow - 1, N_\downarrow; \phi - \frac{1}{N}\Big)|b_{\ell\uparrow}|\Psi_0(N_\uparrow, N_\downarrow; \phi)\rangle|^2. \quad (5)$$

We again set $N_\uparrow = N_\downarrow = N_e/2$ for N_e even, and $N_\uparrow = N_\downarrow + 1 = (N_e + 1)/2$ for N_e odd. Figure 3(a) shows $O_{\rm B}$ as a function of $n_e(= \phi)$. The order parameter has a pronounced enhancement in both the high-density region $(0.7 < n_e < 1)$ and the low-density one $(0 < n_e \lesssim 0.3)$. Figures 3(b) and 3(c) depict the ϕ dependence of $O_{\rm B}$ for $n_e = 6/20$ and $16/20$, respectively. We find a salient growth at $\phi = n_e$ for both densities. We have also observed for $n_e = 6/20$ that the order parameter given by Eq. (5) with $m = -1$ has a peak at $\phi = 1 - n_e$. These results again support the ferromagnetism based on the statistical transmutation.

Finally, we discuss the region $0.6 \lesssim n_e \lesssim 0.7$, where the saturated ferromagnetism is absent in spite of $\phi = n_e$. As a candidate of the competing order, we consider the spin chirality [24] defined by the order parameter $\chi_{\rm ch} = (1/N)\sum_\ell \langle \boldsymbol{S}_\ell \cdot \boldsymbol{S}_{\ell+\hat{y}} \times \boldsymbol{S}_{\ell+\hat{x}}\rangle$, where $\hat{x}$ $(\hat{y})$ is the unit vector along x (y) direction, and $\langle\cdots\rangle$ denotes the expectation value in the ground state. In fact, Nagaoka ferromagnetism in the single-hole case is known to be destroyed by development of the spin chirality in the presence of a perpendicular magnetic field [18]. Along the line $\phi = n_e$, we have confirmed that the spin chirality vanishes when the system exhibits saturated ferromagnetism. On the other hand, the chiral order is indeed developed when the saturated ferromagnetism is absent (not shown).

In summary, we have calculated the total spin of the ground state in the $U \to \infty$ Hubbard model with magnetic flux (ϕ) perpendicular to a square lattice and revealed regions of saturated ferromagnetism. The saturated ferromagnetism at $\phi = n_e$ is argued to be due to formation of spinful composite bosons. Statistical transmutation may therefore play a key role in ferromagnetism in strongly correlated systems, just as it did in fractional quantum Hall effect.

The present mechanism may be relevant to future experiments on an artificial crystal of a square lattice with quantum dots (i.e., a quantum dot superlattice) [25]. A large lattice constant of such a crystal would enable us to observe the magnetic-field effect at a modest magnetic field of a few Tesla. In this situation, the orbital motion rather than the Zeeman effect could be essential for the emergence of ferromagnetism. Another possibility is to induce the effective gauge field internally without an applied magnetic field [26].

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