PHYSICAL REVIEW LETTERS

Nonequilibrium Phase Transition on a Randomly Diluted Lattice

Thomas Vojta and Man Young Lee

Department of Physics, University of Missouri-Rolla, Rolla, Missouri 65409, USA

We show that the interplay between geometric criticality and dynamical fluctuations leads to a novel universality class of the contact process on a randomly diluted lattice. The nonequilibrium phase transition across the percolation threshold of the lattice is characterized by unconventional activated (exponential) dynamical scaling and strong Griffiths effects. We calculate the critical behavior in two and three space dimensions, and we also relate our results to the recently found infinite-randomness fixed point in the disordered one-dimensional contact process.

Nonequilibrium systems can undergo continuous phase transitions between different steady states. These transitions are characterized by collective fluctuations over large distances and long times similar to the behavior of equilibrium critical points. Examples can be found in population dynamics and epidemics, chemical reactions, growing surfaces, and in granular flow and traffic jams (for recent reviews see, e.g., Refs. [1–6]).

If a nonequilibrium process is defined on a randomly diluted spatial lattice, its dynamical fluctuations coexist with geometric fluctuations. Site or bond dilution defines a percolation problem for the lattice with a geometric phase transition at the percolation threshold [7]. In this Letter we address the question of how the interplay between geometric criticality due to percolation and dynamical fluctuations of the nonequilibrium process influences the properties of the phase transition.

Our starting point is the contact process [8], a prototypical system exhibiting a nonequilibrium phase transition. It is defined on a d-dimensional hypercubic lattice ($d \geq 2$). Each site can be active (occupied by a particle) or inactive (empty). In the course of the time evolution, active sites infect their neighbors, or they spontaneously become inactive. Specifically, the dynamics is given by a continuous-time Markov process during which particles are created at empty sites at a rate $\lambda n/(2d)$ where n is the number of active nearest neighbor sites. Particles are annihilated at unit rate. For small birth rate λ, annihilation dominates, and the absorbing state without any particles is the only steady state (inactive phase). For large birth rate λ, there is a steady state with finite particle density (active phase). The two phases are separated by a nonequilibrium phase transition in the directed percolation [9,10] universality class at some $\lambda = \lambda_c^0$.

We introduce quenched site dilution [11] by randomly removing lattice sites with probability p. The resulting phase diagram of the site-diluted contact process is sketched in Fig. 1. For small impurity concentrations below the percolation threshold of the lattice, $p < p_c$, the active phase survives, but the critical birth rate increases with p (to compensate for the missing neighbors). Right at

the percolation threshold the active phase survives on the infinite percolation cluster for $\lambda > \lambda_*$. The (multi) critical birthrate λ_* must be smaller than the critical birthrate of the one-dimensional (1D) contact process because the critical percolation cluster is connected, infinitely extended, and its fractal dimension is $D_f > 1$. For $p > p_c$, no active phase can exist because the lattice consists of disconnected clusters of finite size that do not support a steady state density of active sites.

The contact process on a site-diluted lattice therefore has two nonequilibrium phase transitions, separated by a multicritical point. For $p < p_c$, the transition (marked by "a" in Fig. 1) is expected to be in the universality class of the generic disordered contact process [13–15] which has reattracted considerable attention recently [16,17]. In contrast, the phase transition across the percolation threshold p_c of the lattice for $\lambda > \lambda_*$ (transition "b" in Fig. 1) has received much less attention.

In this Letter, we show that the interplay between geometric criticality and dynamic fluctuations leads to a novel universality class for this nonequilibrium phase transition. Even though the transition is driven entirely by the geometry of the lattice, the dynamical fluctuations of the contact
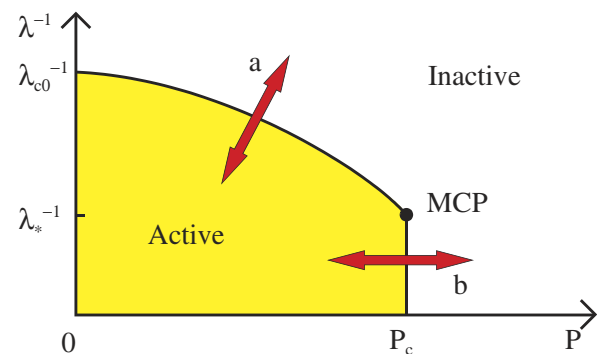

FIG. 1 (color online). Schematic phase diagram of a site-diluted contact process as a function of impurity concentration p and birth rate λ. There is a multicritical point at $p = p_c$ and $\lambda = \lambda_*$. The phase transition (b) across the percolation threshold of the lattice is the topic of this Letter.

035701-1

process enhance the singularities in all quantities involving dynamic correlations. Our results can be summarized as follows. The dynamical scaling is not of conventional power-law form but activated, i.e., the relation between correlation length $\xi_\perp$ and correlation time $\xi_\parallel$ is exponential,

$$\ln \xi_\parallel \sim \xi_\perp^\psi, \tag{1}$$

with the critical exponent ψ being equal to the fractal dimension of the critical percolation cluster, $\psi = D_f$. As a result, the long-time decay of the density ρ of active sites at $p = p_c$ is ultraslow,

$$\rho(t) \sim [\ln(t/t_0)]^{-\bar{\delta}}. \tag{2}$$

The exponent $\bar{\delta} = \beta_c/(\nu_c D_f)$ is determined by D_f together with the order parameter and correlation length exponents, β_c and ν_c, of the lattice percolation transition [18]. In contrast to the enhanced dynamical singularities, the exponents of static quantities like the steady state density ρ_{st} and the spatial correlation length $\xi_\perp$ are identical the corresponding lattice percolation exponents,

$$\rho_{\mathrm{st}}(p) \sim |\Delta|^{\beta_c} \qquad (\Delta < 0), \tag{3}$$

$$\xi_\perp \sim |\Delta|^{\nu_c}. \tag{4}$$

where $\Delta = p - p_c$ measures the distance from the percolation threshold. Off criticality, i.e., away from the percolation threshold, we find strong Griffiths effects [19] characterized by a nonexponential density decay [13],

$$\rho(t) \sim (t/t_0)^{-d/z'} \qquad (p > p_c), \tag{5}$$

$$\rho(t) - \rho_{\mathrm{st}} \sim e^{-[(d/z'')\ln(t/t_0)]^{1-1/d}} \qquad (p < p_c), \tag{6}$$

where the nonuniversal exponents z' and z'' diverge as $z', z'' \sim \xi_\perp^{D_f}$ for $p \to p_c$. In the remainder of the Letter we sketch the derivation of these results and calculate exponent values and additional observables. We also relate our results to the disordered 1D contact process [16,17] and to the diluted quantum Ising model [20].

Let us start by considering the steady state density ρ_{st} of active sites (i.e., the order parameter of the transition). A nonzero steady state density can only develop on the infinite percolation cluster; finite clusters do not contribute because they eventually go into the inactive state via a rare fluctuation. For $\lambda > \lambda_c$, the infinite cluster is in the active phase. The total steady state density is proportional to the number of sites in the infinite cluster, $\rho_{\mathrm{st}} \sim P_\infty(p) \sim (p_c - p)^{\beta_c}$. The order parameter exponent of the nonequilibrium transition is therefore identical to that of the lattice percolation transition, $\beta = \beta_c$, as stated in (3). To determine the spatial correlation length $\xi_\perp$ we note that the correlations of the contact process cannot extend beyond the connectedness length ξ_c of the percolating lattice because sites on different percolation clusters are decoupled. On the other hand, for $\lambda > \lambda_*$, all sites on the

same cluster are strongly correlated in space even though they collectively fluctuate in time. We thus conclude $\xi \sim \xi_c$ and $\nu = \nu_c$ in agreement with (4).

We now study the time dependence of the density $\rho(t)$ of active sites, starting from a completely active lattice. We first consider the contact process on a single percolation cluster of finite size (number of sites) s. For $\lambda > \lambda_*$ such a cluster is locally in the active phase. It therefore has a metastable state with a nonzero density of active sites. This metastable state can decay into the inactive state only via a rare collective fluctuation involving all sites of the cluster. The probability for such a rare event decreases exponentially with the size s of the cluster. Therefore, the lifetime t_s of the metastable active state on a cluster increases exponentially with its size s,

$$t_s(s) \sim t_0 e^{A(\lambda)s}, \tag{7}$$

where t_0 is some microscopic time scale. The prefactor $A(\lambda)$ vanishes at the multicritical point, $A(\lambda_*) = 0$, and increases with increasing λ. The number of sites s of a percolation cluster is connected to its linear size R_s via $s \sim R_s^{D_f}$. Therefore, (7) establishes the exponential relation between length and time scales, $\ln t_s \sim R_s^{D_f}$ leading to activated dynamical scaling (1).

After having analyzed a single cluster we turn to the full percolation problem. From classical percolation theory, we know that close to p_c the number n_s of occupied clusters of size s per lattice site (excluding the infinite cluster for $p < p_c$) obeys the scaling form

$$n_s(\Delta) = s^{-\tau_c} f(s\Delta^{1/\sigma_c}). \tag{8}$$

The scaling function $f(x)$ behaves as

$$f(x) \sim \exp(-B_1 x) \qquad (p > p_c), \tag{9}$$

$$f(x) = \mathrm{const} \qquad (p = p_c), \tag{10}$$

$$f(x) \sim \exp[-(B_2 x)^{1-1/d}] \qquad (p < p_c), \tag{11}$$

where B_1, B_2 are constants. The exponents τ_c and σ_c determine all critical exponents of the percolation transition of the lattice including the correlation length exponent $\nu_c = (\tau_c - 1)/(d\sigma_c)$, the order parameter exponent $\beta_c = (\tau_c - 2)/\sigma_c$, and the fractal dimension $D_f = d/(\tau_c - 1)$ of the percolating cluster [7].

In order to obtain the total density of active sites for the contact process on the diluted lattice, we sum the number of active sites over all percolation clusters. Combining the cluster size distribution (8) with the lifetime of the metastable active state (7) leads to

$$\rho(t, \Delta) \sim \int ds\, s\, n_s(\Delta) \exp[-t/t_s(s)]. \tag{12}$$

Right at the percolation threshold, this reduces to

$$\rho(t, 0) \sim \int ds\, s^{1-\tau_c} \exp[-(t/t_0 e^{As})]. \tag{13}$$

The leading behavior of this integral can be found by noticing that only islands with size $s > s_{\min}(t) = A^{-1}\ln(t/t_0)$ contribute at time t. The critical long-time dependence of the total density is thus given by

$$\rho(t, 0) \sim [\ln(t/t_0)]^{2-\tau_c} \qquad (p = p_c). \qquad (14)$$

This completes the derivation of (2) with the critical exponent $\bar{\delta}$ given by $\bar{\delta} = \tau_c - 2 = \beta_c/(\nu_c D_f)$, in agreement with general scaling arguments [16,17].

We now consider the behavior of the density off criticality. In the inactive phase, $p > p_c$, the time dependence of the density is given by

$$\rho(t, \Delta) \sim \int ds\, s^{1-\tau_c} \exp[-B_1 s\Delta^{1/\sigma_c} - (t/t_0 e^{As})]. \quad (15)$$

For long times, the leading behavior of the integral can be calculated using the saddle-point method, giving

$$\rho(t, \Delta) \sim t^{-(B_1/A)\Delta^{1/\sigma_c}} \qquad (p > p_c), \qquad (16)$$

equivalent to (5). The nonuniversal exponent z' is given by $z' = (A d/B_1)\Delta^{-1/\sigma_c} \sim \xi_\perp^{D_f}$.

In the active phase, $p < p_c$, there is a nonzero steady state density $\rho_{\rm st}$ coming from the infinite percolation cluster. However, the approach of the density towards this value is still determined by the slow decay of the metastable states of the finite percolation clusters

$$\rho(t, \Delta) - \rho_{\rm st}(\Delta) \sim \int ds\, s^{1-\tau_c} \exp[-(B_2 s|\Delta|^{1/\sigma_c})^{1-1/d}$$
$$- (t/t_0 e^{As})]. \qquad (17)$$

Using the saddle-point point method to calculate the leading long-time behavior gives (for $p < p_c$),

$$\rho(t, \Delta) - \rho_{\rm st}(\Delta) \sim e^{-[(B_2/A)|\Delta|^{1/\sigma_c}\ln(t/t_0)]^{1-1/d}}. \qquad (18)$$

This completes the derivation of (6) with $z'' = (A d/B_2)|\Delta|^{-1/\sigma_c} \sim \xi_\perp^{D_f}$. The nonexponential off-critical relaxation of the density (5) and (6) is characteristic of a Griffiths region in the contact process [13,19]. We also point out that time and spatial correlation length enter these equations in the form of the combination $\ln(t)/\xi_\perp^{D_f}$, again characteristic of activated scaling.

We now turn to the influence of an external source field h that describes spontaneous particle creation at a rate h at each lattice site. To determine the steady state density as a function of h we again start by considering a single percolation cluster of size s. For $\lambda > \lambda_*$, the cluster is active if at least one particle has been spontaneously created on one of the s sites within the lifetime $t_s(s) = t_0 e^{As}$. For small h, the average number of particles created on a cluster of size s within time t_s is $M_s(h) = hst_s = hst_0 e^{As}$. If $M_s > 1$, the cluster is (almost) always active. If $M_s < 1$, it is active with a probability proportional to M_s. The total steady state density is obtained by summing over all clusters

$$\rho_{\rm st}(h, \Delta) \sim \int ds\, s n_s(\Delta) \min[1, M_s(h)]. \qquad (19)$$

Evaluating this integral analogously to the time-dependent density (12) yields, for small fields h,

$$\rho_{\rm st}(h) \sim [\ln(h_0/h)]^{-\bar{\delta}} \qquad (p = p_c), \qquad (20)$$

$$\rho_{\rm st}(h) \sim (h/h_0)^{d/z'} \qquad (p > p_c), \qquad (21)$$

$$\rho_{\rm st}(h) \sim \exp[-(d/z'')\ln(h_0/h)]^{1-1/d} \qquad (p < p_c), \quad (22)$$

where $h_0 \sim 1/t_0$. At $p = p_c$, the relation between density and field is logarithmic, as expected from activated scaling. Off criticality, we find strong Griffiths effects similar to those in the time dependence of the density.

The above results can also be derived from a scaling theory. In the active phase, the density is proportional to the number of sites in the infinite percolation cluster. Thus, its scale dimension must be β_c/ν_c. Time must enter via the scaling combination $\ln(t)b^{D_f}$ reflecting the exponential dependence of the lifetime (7) on the cluster size. The field h, being a rate, scales like inverse time. We therefore obtain the following scaling form:

$$\rho[\Delta, \ln(t), \ln(1/h)] = b^{\beta_c/\nu_c}\rho[\Delta b^{-1/\nu_c}, \ln(t)b^\psi, \ln(1/h)b^\psi], \qquad (23)$$

where b is an arbitrary (length) scale factor and $\psi = D_f$. This form is consistent with all our explicit results.

All critical exponents of the nonequilibrium phase transition are determined by the classical percolation exponents of the lattice. In two space dimensions, their values are known exactly and in three dimensions they are known numerically with high accuracy [7]. Table I shows numerical exponent values for these cases.

We also briefly comment on the early time behavior. For $\lambda > \lambda_*$ each percolation cluster is locally in the active phase. Starting from a single active seed, the cloud of active sites thus initially grows ballistically, i.e., the radius of the cloud grows linearly with time, until a metastable state is reached in which it covers the entire percolation cluster. The time required for this initial spreading on an island of size s is $t_i(s) \sim R_s \sim s^{1/D_f}$. As discussed above, the metastable state decays only at the much larger time scale $t_s(s) \sim e^{As}$. We thus arrive at the somewhat surprising conclusion that the early time behavior of the contact process on our diluted lattice is much faster than the logarithmically slow long-time decay of the density.

In the remaining paragraphs, we discuss the generality of our results, compare them to the transition in the diluted

TABLE I. Critical exponents of the nonequilibrium phase transition at $p = p_c$ in two and three space dimensions.

Exponent	$d = 2$	$d = 3$
$\beta = \beta_c$	5/36	0.417
$\nu = \nu_c$	4/3	0.875
$\psi = D_f = d - \beta_c/\nu_c$	91/48	2.523
$\bar{\delta} = \beta_c/(\nu_c D_f)$	5/91	0.188

quantum Ising model [20] and to the recently found infinite-randomness critical point in a random 1D contact process [16,17]. We also compare to a general classification of phase transitions with quenched disorder [21].

The logarithmic time and field dependencies (2) and (20) at the nonequilibrium phase transition at $p = p_c$, as well as the strong Griffiths effects in its vicinity, are the direct result of combining the spectrum of percolation cluster sizes (8) with the exponential dependence (7) of the lifetime on the cluster size. We therefore expect similar behavior in other diluted equilibrium or nonequilibrium systems that share this exponential relation between length and time scales. One example is the diluted transverse field Ising model [20]. In this system, the energy gap of a cluster decreases exponentially with its size. As a result, the scaling behavior at the quantum phase transition across the percolation threshold of the lattice is very similar to the one found in this Letter.

Recently, the critical point of the 1D contact process with spatial disorder was found to be of infinite-randomness type [16,17]. Analogous behavior is expected for the generic disordered directed percolation transition in higher dimensions, e.g., the transition at $p < p_c$ in the site-diluted contact process (transition a in Fig. 1). Our critical point at the percolation threshold shares some characteristics with these infinite-randomness critical points, notably the exponential relation between correlation length and time as well as the logarithmically slow decay of the total density. However, it belongs to a different universality class with novel critical exponents. Moreover, the early time behavior is different (logarithmically slow at the generic infinite-randomness critical point but of power-law type at our transition).

Lastly, we point out that our results are in agreement with a general classification of phase transitions with quenched disorder (and short-range interactions) according to the effective dimensionality d_{eff} of the droplets or clusters [21]. Three cases can be distinguished: (i) If the clusters are below the lower critical dimension of the problem, $d_{\text{eff}} < d_c^-$, the critical behavior is of conventional power-law type and the Griffiths effects are exponentially weak. (ii) If $d_{\text{eff}} = d_c^-$, the critical point shows activated scaling accompanied by strong, power-law Griffiths effects. This case is realized in random transverse field Ising magnets [20,22] as well as in our diluted contact process. (iii) If $d_{\text{eff}} > d_c^-$, the phase transition is smeared because locally ordered clusters can undergo the phase transition independently from the bulk. This occurs, e.g., for some metallic quantum magnets [23] or for the contact process with extended defects [24,25].

In conclusion, we have shown that the contact process on a diluted lattice has unusual properties. The interplay between geometric criticality and dynamical fluctuations leads to a novel universality class with activated scaling and ultraslow dynamics. Interestingly, despite its ubiquity in theory, experimental observations of directed percolation scaling [26] are very rare. Our results suggest that

peculiar disorder effects may be responsible for this in at least some of the experiments.

[1] B. Schmittmann and R. K. P. Zia, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic, New York, 1995). Vol. 17, p. 1.

[2] J. Marro and R. Dickman, *Nonequilibrium Phase Transitions in Lattice Models* (Cambridge University Press, Cambridge, England, 1996).

[3] R. Dickman, in *Nonequilibrium Statistical Mechanics in One Dimension*, edited by V. Privman (Cambridge University Press, Cambridge, England, 1997), p. 51.

[4] H. Hinrichsen, Adv. Phys. **49**, 815 (2000).

[5] G. Odor, Rev. Mod. Phys. **76**, 663 (2004).

[6] U. C. Täuber, M. Howard, and B. P. Vollmayr-Lee, J. Phys. A **38**, R79 (2005).

[7] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor & Francis, London, 1994).

[8] T. E. Harris, Ann. Probab. **2**, 969 (1974).

[9] P. Grassberger and A. de la Torre, Ann. Phys. (N.Y.) **122**, 373 (1979).

[10] H. K. Janssen, Z. Phys. B **42**, 151 (1981); P. Grassberger, Z. Phys. B **47**, 365 (1982).

[11] Our percolating lattice is externally given. This differs from the general epidemic process with immunization where the nonequilibrium process itself dynamically generates a percolation cluster of inaccesible sites [12].

[12] P. Grassberger, Math. Biosci. **63**, 157 (1982); J. Cardy, J. Phys. A **16**, L709 (1983); H. K. Jannssen, Z. Phys. B **58**, 311 (1985).

[13] A. J. Noest, Phys. Rev. Lett. **57**, 90 (1986); Phys. Rev. B **38**, 2715 (1988).

[14] H. K. Janssen, Phys. Rev. E **55**, 6253 (1997).

[15] A. G. Moreira and R. Dickman, Phys. Rev. E **54**, R3090 (1996); R. Dickman and A. G. Moreira, *ibid.* **57**, 1263 (1998).

[16] J. Hooyberghs, F. Igloi, and C. Vanderzande, Phys. Rev. Lett. **90**, 100601 (2003); Phys. Rev. E **69**, 066140 (2004).

[17] T. Vojta and M. Dickison, Phys. Rev. E **72**, 036126 (2005).

[18] In the following, lattice percolation exponents carry a subscript c while the exponents of the nonequilibrium transition of the contact process do not carry a subscript.

[19] R. B. Griffiths, Phys. Rev. Lett. **23**, 17 (1969).

[20] T. Senthil and S. Sachdev, Phys. Rev. Lett. **77**, 5292 (1996).

[21] T. Vojta and J. Schmalian, Phys. Rev. B **72**, 045438 (2005).

[22] D. S. Fisher, Phys. Rev. B **51**, 6411 (1995).

[23] T. Vojta, Phys. Rev. Lett. **90**, 107202 (2003).

[24] T. Vojta, Phys. Rev. E **70**, 026108 (2004).

[25] M. Dickison and T. Vojta, J. Phys. A **38**, 1199 (2005).

[26] H. Hinrichsen, Braz. J. Phys. **30**, 69 (2000).