

Model for Current-Driven Edge-Localized Modes

C. G. Gimblett,* R. J. Hastie, and P. Helander

EURATOM/UKAEA Fusion Association, Culham Science Centre, Abingdon, Oxon, OX14 3DB, United Kingdom[†]
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Edge-localized modes (ELMs) are cyclic disturbances in the outer region of tokamak plasmas that are influential in determining present and future tokamak performance. In this Letter, we outline an approach to modeling ELMs in which we envisage toroidal peeling modes initiating a Taylor relaxation [Phys. Rev. Lett. **33**, 1139 (1974)] of a tokamak outer region plasma. Relaxation produces a peeling destabilizing flattened edge current profile and a stabilizing plasma-vacuum current sheet; the balance between the two determines the radial extent of the relaxed region. The model can be used to predict the energy losses due to an ELM and reproduces experimentally observed variations with edge safety factor and plasma collisionality. There is an intrinsic “deterministic scatter” in the model that also accords with observation.

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Theoretical models of edge-localized modes (ELMs) have widely been based upon the assumed presence of “ballooning” and/or “peeling” ideal magnetohydrodynamic instabilities [1,2]. ELMs are an important factor in determining global tokamak performance, and the nature of ELM power loss deposition presents a challenge to the design of the proposed International Thermonuclear Experimental Reactor divertor [3]. In this context, systematic integrated codes that incorporate specific ELM models have been developed [4–7]. Here we describe an approach to modeling ELMs in which we envisage edge current density (J_a) driven ideal toroidal peeling modes initiating a Taylor relaxation [8] of a tokamak outer region plasma. The model therefore assumes that the plasma is *below* the ballooning stability limit and is perhaps best suited to describe the smaller, more frequent, “type III” ELMs which occur soon after the plasma shifts from low to high confinement (the L-H transition) [2,9].

When we adopt the large aspect ratio expansion of a tokamak, the stability criterion for ideal toroidal peeling modes is given by [10]

$$\alpha \left\{ \frac{r}{R_0} \left(1 - \frac{1}{q^2} \right) + s \frac{d\Delta_{\text{sh}}}{dr} - \mathcal{F}_t \frac{sR_0}{2r} \right\} > s q R_0 \frac{J_a}{B_0}, \quad (1)$$

where $\alpha = -2(\mu_0 R_0 q^2 / B_0^2) dp/dr$ (with p the pressure), $s = (r/q) dq/dr$ is the magnetic shear (with q the safety factor), Δ_{sh} is the Shafranov shift, and \mathcal{F}_t is a quantity related to the fraction of trapped particles. The pressure gradient terms on the left-hand side of Eq. (1) represent the stabilizing effect of favorable average curvature (Mercier), a stabilizing contribution from the Pfirsch-Schlüter currents ($\propto d\Delta_{\text{sh}}/dr$), and a destabilizing contribution from the (\mathcal{F}_t dependent) “bootstrap” current. The effect of separatrix geometry on peeling is not taken into account in Eq. (1); this challenging issue has recently been explored computationally [11].

We may expect that, when the toroidal peeling stability boundary is crossed, a rapid process of energy release occurs, producing the above mentioned post-ELM Taylor

state. This force-free state in a tokamak has a flattened toroidal current profile, and we assume that the pressure in the edge is entirely lost. At first sight, this would appear to generate an even more unstable situation for peeling modes, as the stabilizing left-hand side of Eq. (1) disappears with the pressure, and the destabilizing right-hand side would, in general, increase, as a flattening of a conventional monotonically decreasing current density profile increases the edge J_a . However, it is known that, assuming the relaxation process occurs quickly compared with current diffusion times, a relaxed plasma-vacuum system generally possesses a skin current distribution at its interfaces [12]. We show that current sheets generated in edge tokamak relaxations generally have a stabilizing effect on peeling modes. In fact, if δ is the width of the edge relaxation zone, then the magnitude of the increase of destabilizing edge current density scales as δ , while the magnitude of the stabilizing sheet current produced scales as δ^2 . Consequently, infinitesimal relaxation widths would produce a post-ELM state which is still peeling unstable. However, above a critical width, the post-ELM Taylor relaxed state can be stable to all possible peeling modes. We use this fact to calculate the ELM relaxation width.

We describe an outline of the analysis by first noting that, once edge pressure gradients are removed, toroidal coupling is absent, and the peeling mode can be treated in the cylindrical approximation (at least in the large aspect ratio limit). Stability of the post-ELM state, then, is governed by the well-known marginal equation of toroidal force balance [13]

$$\frac{d}{dr} \left(r \frac{d\psi}{dr} \right) - \frac{m^2 \psi}{r} = \frac{m}{F} \mu_0 \frac{dJ}{dr} \psi, \quad (2)$$

where ψ is the perturbed poloidal flux, J is the equilibrium current density, and $F = (B_\theta/r)(m - nq)$, with m, n the poloidal and toroidal wave numbers, respectively, of the perturbation. Equation (2) holds everywhere in the plasma and vacuum regions but is subject to boundary conditions both at the plasma/vacuum (\mathcal{P}/\mathcal{V} , $r = a$) interface and at

the internal plasma radius that defines the inner boundary of the relaxed region, $r = r_E$ (Taylor relaxation is thus assumed to occur within the annulus $r_E < r < a$). The boundary conditions to be applied at the radial stations $r = a$ and $r = r_E$ correspond to demanding that the tangential stress on the perturbed interfaces be continuous and that the perturbed interfaces remain flux surfaces. In the presence of a surface skin current, the latter reduces to demanding that ψ/F also be continuous (full details of the analysis will be given in a future publication).

Before showing the model equations that result, we define four quantities which arise in the analysis that represent physically relevant equilibrium and perturbation quantities in the system: (i) $I = \mu_0 R_0 J / B_0 = (1/r)d/dr(r^2/q)$, the dimensionless toroidal current density related to the safety factor q ; (ii) $\Delta = (1/q - n/m)$, a dimensionless measure of the “distance” between a radial position and the “resonance” where $m = nq$ (for peeling modes, Δ is characteristically a small positive number, so a resonance occurs just outside the \mathcal{P}/\mathcal{V} interface); (iii) $\mathcal{K} = \mu_0 I_s R_0 / (aB_0) = [[1/q]]$, where $[[1/q]]$ denotes a jump across the radial station at which I_s is the surface skin current; and (iv) $\Delta' = [[(r/\psi)d\psi/dr]]$, the jump in the perturbed poloidal flux radial derivative which is central to MHD stability analysis. [The notation Δ' is standard, and it should be stressed that Δ' is not related to the Δ defined in (ii).]

We denote the \mathcal{P}/\mathcal{V} and r_E radii with a, E subscripts, respectively, and, after some algebra, the boundary conditions produced by the sheet currents at $r = a$ and $r = r_E$ are found to be

$$\Delta_a [\Delta_a \Delta'_a + I_a] + \mathcal{K}_a \left[(\mathcal{K}_a - 2\Delta_a)(\Delta'_a + m - 1) + \frac{2n}{m} - I_a \right] = 0, \quad (3)$$

$$\Delta_{E-} [\Delta_{E-} \Delta'_{E-} + I_{E-} - I_{E+}] + \mathcal{K}_E \left[(\mathcal{K}_E + 2\Delta_{E-}) \times (\Delta'_{E-} + m + 1) + \frac{2n}{m} - I_{E+} \right] = 0. \quad (4)$$

[We can show that the left-hand side of Eq. (3) is proportional to $-\delta W$, the ideal MHD energy perturbation [14].]

To complete the mathematical model to be solved, we must connect Δ'_a to Δ'_{E-} across the relaxed region $r_E < r < a$ and, thus, link the radial stations at $r = a$ [Eq. (3)] and $r = r_E$ [Eq. (4)]. This is a straightforward procedure that uses the solution to Eq. (2) in ideal subintervals [15]; we give an example application below, after first describing the calculation of the post-ELM state.

The original Taylor relaxation calculation [8] consisted of a constrained minimization of the magnetic field energy. The relevant conserved quantities for a highly conducting plasma were the total toroidal magnetic flux Ψ_z and the global helicity K of the magnetic field, $K = \int_V \mathbf{A} \cdot \mathbf{B} dV$

(with \mathbf{A} the magnetic vector potential, $\mathbf{B} = \nabla \times \mathbf{A}$). Within the cylindrical tokamak ordering, Ψ_z conservation is implicit. A gauge-invariant definition of the magnetic helicity K must take account of the annular, multiply connected topology [16] of the region in question. Accordingly, for our geometry K reduces to $K = \int_{r_E}^a (r/q)(r^2 - r_E^2) dr$.

As we are dealing with an *annular* plasma region and, hence, two cylindrical boundaries, it will be necessary to invoke a second invariant of the system to determine a final state. In the same spirit of the Taylor hypothesis that the global helicity is the relevant conserved quantity for a highly conducting tokamak plasma, the natural second quantity to be conserved throughout the relaxation process in our geometry is the total annular poloidal magnetic flux $\Psi_\theta = \int_{r_E}^a (r/q) dr$.

Combining these considerations, our extended relaxation problem can be formulated as that of finding a minimization of the poloidal magnetic energy W_θ , subject to conservation of both K and Ψ_θ . Formally, we require variations in the functional

$$W_\theta - \lambda_1 K - \lambda_2 \Psi_\theta = \int_{r_E}^a \left[\frac{r^3}{q^2} - \lambda_1 \frac{r}{q} (r^2 - r_E^2) - \lambda_2 \frac{r}{q} \right] dr \quad (5)$$

to be stationary (with $\lambda_{1,2}$ Lagrangian multipliers). This problem has the solution $q^f(r) = r^2 / (Cr^2 + D)$ in $r_E < r < a$. Here the superscript f denotes the final relaxed profile, and C, D are constants to be determined; in fact, the q^f profile corresponds to uniform toroidal equilibrium current density. We finally have to specify the post-ELM state of the external vacuum fields. In general, this must take account of the interaction of the plasma with the externally imposed experimental circuit conditions, and here we take the vacuum poloidal flux to be unchanged and, hence, take the total plasma current to be held constant. Note that the formalism has no fitted parameters and gives a uniquely defined final state once the initial q profile has been specified.

When edge peeling marginality, Eq. (1), is reached and, subsequently, edge pressure gradient is lost, then Eqs. (3) and (4) become the equations governing stability of the post-ELM state. It is then required to find a relaxed state that is stable to *all* possible peeling modes. We now have all the necessary parts of the model to perform an example calculation.

To produce some illustrative results, we investigate an initial simple parabolic safety factor profile

$$q^i = q_0 + (q_a - q_0)r^2, \quad 0 \leq r \leq 1. \quad (6)$$

We should now use this q profile to find solutions to Eq. (2) and, hence, connect the $\Delta'_{a,E}$ of Eqs. (3) and (4). In fact, the relaxed region ($dJ/dr = 0$) gives $\psi \propto r^{\pm m}$, while the vacuum region has $\psi \propto r^{-m}$. In this example, we are primarily

interested in high m peeling modes so we can take the solution of Eq. (2) in $0 < r < r_E$ to be $\propto r^m$ to good approximation, and it follows that

$$\Delta'_E = -2m(\Delta'_a + 2m)/(g\Delta'_a + 2m), \quad (7)$$

where $g = 1 - (r_E/a)^{2m}$.

For a given (q_0, q_a) in Eq. (6), Eqs. (3) and (4) (with $\mathcal{K}_{a,E} = 0$) give a sequence of (m, n) pairs (concentrating at $m = nq_a$) for which the initial profile is peeling unstable [left-hand side of Eq. (3) $\propto -\delta W > 0$]. We introduce a normalized ELM width $d_E = (a - r_E)/a$, and then, for each unstable (m, n) pair, we increase d_E in Eqs. (3) and (4) (using the relaxed final state in $r_E < r < a$) until $\delta W = 0$, and peeling marginality is achieved. It is then natural to assume that the model ELM width for any (q_0, q_a) corresponds to the largest marginal $d_E (= d_E(\max))$ obtained over all the initially unstable (m, n) pairs. Next we may ask how the $d_E(\max)$ values vary as the initial equilibrium is varied.

Figure 1 shows the result of such a calculation, and we have plotted $d_E(\max)$ for $q_0 = 1$ and a range of q_a . Note that a feature of this plot is the “deterministic” scatter of the results; this behavior can be traced to intrinsic variability in the rational (m/n) approximation to q_ν in the quantity $\Delta_a = (1/q_\nu - n/m)$. Note the concentration of large d_E values to the near left-hand side of integer q_a (indeed, for $q_a < 3$, d_E rises to ~ 0.5). Figure 2 plots the n values for which the maximal mode exists, and we can see that the large d_E excursions of Fig. 1 correspond to low n modes. To avoid confusion, we note that this does not imply that the ELMs themselves have low mode numbers, merely that the post-ELM state should be stable for both small as well as large n modes.

For small ELM widths, it proves possible to expand the entire set of equations determining the system and derive an analytic expression for $d_E(\max)$. An expansion of the post-ELM equilibrium gives $\mathcal{K}_a \propto -d_E^2$. Inspection of Eq. (7) shows that, when $d_E \gg 1/m$, Eqs. (3) and (4) decouple and (3) gives a quadratic in d_E^2 . We can then

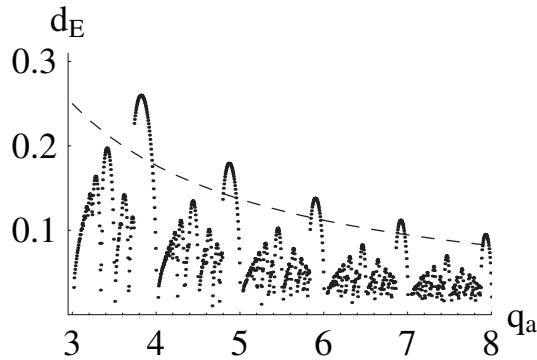


FIG. 1. The maximal marginal d_E for the initial parabolic q profile of Eq. (6), plotted against q_a ($q_0 = 1$). An analytical approximation for $n = 1$ modes is given as the dashed curve [see Eq. (8)].

maximize d_E in this quadratic by varying Δ_a (taking it to be a continuous variable). On further taking the edge current density to be small, the maximal relaxation width within this expansion is given by

$$d_E^2(\max) = -\frac{3}{4n} \frac{I_a^2}{(aI'_a)} \quad (8)$$

(here $'$ denotes d/dr). For the parabolic q profile of Eq. (6), we find $d_E(\max) = \sqrt{3q_0/8nq_a(q_a - q_0)}$, and this approximation to $d_E(\max)$ for $q_0 = 1$, $n = 1$ is shown in Fig. 1 as the dashed curve.

If we combine the results of Fig. 1 with the value of the critical pressure gradient for the onset of relaxation as given by Eq. (1), we can predict the experimentally measured values of ELM energy loss (ΔW_{ELM}) as a fraction of the total plasma energy (W_{PED}) calculated, assuming an equilibrium pressure equal to the pedestal value everywhere in the core. As an example, we take a characteristic mega-ampere spherical tokamak (MAST) plasma [17] with a highly collisional edge [$\mathcal{F}_t = 0$ in Eq. (1)] and plot $\Delta W_{\text{ELM}}/W_{\text{PED}}$ against q_a in Fig. 3.

Typically, $\Delta W_{\text{ELM}}/W_{\text{PED}}$ is a few percent, in accord with the observations [17]. Figure 3 shows that (at least for a parabolic q profile) there is a trend of ELM losses decreasing with increasing q_a . Further, consideration of Eq. (1) indicates that the critical pressure gradient (α) for toroidal peeling modes increases with \mathcal{F}_t . This quantity decreases with collisionality, so we may expect increased ELM losses as the plasma edge becomes less collisional. These trends have been reported (at least for type I ELMs) on the Joint European Torus tokamak [18].

In summary, we have considered a new model for ELM instabilities that hypothesizes an edge Taylor relaxation initiated by toroidal peeling modes. As this nonlinear process proceeds radially inwards, it will leave in its wake a Taylor relaxed state, which, for conventional Tokamak ordering, implies a flattened toroidal current density. This in itself would further destabilize peeling modes; however, a stabilizing edge skin current is *also* formed by the relaxation, and this can lead to an outer

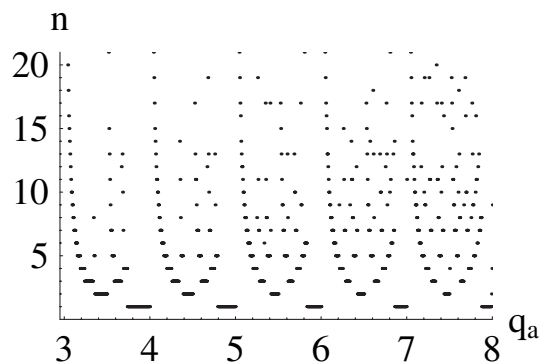


FIG. 2. The toroidal mode number n which gives the d_E values of Fig. 1.

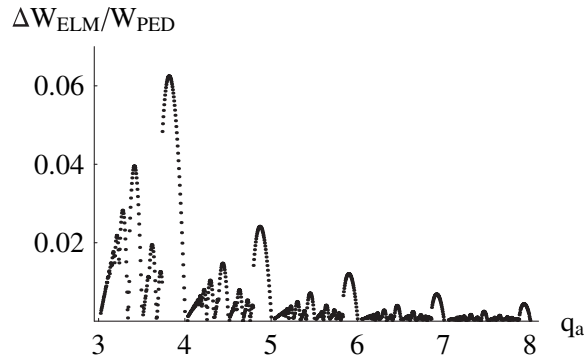


FIG. 3. Normalized ELM energy loss $\Delta W_{\text{ELM}}/W_{\text{PED}}$ of the initial q profile of Eq. (6) plotted against q_a ($q_0 = 1$). The aspect ratio (1.5) is MAST-like [17], and a fully collisional edge plasma is assumed [$\mathcal{F}_t = 0$ in Eq. (1)].

annular region that is stable to all peeling modes. The predicted ELM widths, energy losses, and their natural scatter are, in general, in accord with experimental observations as are the dependence on edge safety factor and collisionality.

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*Electronic address: chris.gimblett@ukaea.org.uk

†Electronic address: www.fusion.org.uk

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