α Effect in a Family of Chaotic Flows

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We perform numerical experiments to calculate the kinematic α effect for a family of maximally helical, chaotic flows with a range of correlation times. We find that the value of α depends on the structure of the flow, on its correlation time and on the magnetic Reynolds number in a nontrivial way. Furthermore, it seems that there is no clear relation between α and the helicity of the flow, contrary to what is often assumed for the parametrization of mean-field dynamo models.

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Cosmic magnetic fields are observed on a vast range of spatial scales in almost all astrophysical objects from planets and stars to galaxies. A fundamental question of astrophysics is then the origin of these fields. Dynamo theory explains the generation and maintenance of magnetic fields through the inductive motions of an electrically conducting fluid. In its simplest form, the theory is derived in the kinematic regime, where the back-reaction of the magnetic field on the velocity via the Lorentz force is ignored, and the induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{U} \times \boldsymbol{B}) + \frac{1}{Rm}\nabla^2 \boldsymbol{B}, \qquad (1)$$

governing the evolution of the magnetic field ($\boldsymbol{B}$) for a prescribed velocity field ($\boldsymbol{U}$), is solved in isolation. Here the magnetic Reynolds number Rm is the nondimensional measure of advection to diffusion; typically Rm is extremely large in astrophysics.

Traditionally, dynamo theory distinguishes between the study of small-scale dynamos, in which magnetic fields are generated on the spatial scale of the inductive motions or smaller, and large-scale dynamos, in which fields are generated on a length scale much larger than those of the velocities. Here we are concerned with the latter, which are often studied within the framework of mean-field electrodynamics [1]. This theory, which is similar in spirit to closure models employed in purely hydrodynamic turbulence, derives an evolution equation for the large-scale field involving transport coefficients that depend on the properties of the small-scale velocity and magnetic fields. The mean-field ansatz decomposes the magnetic and velocity fields into mean and fluctuating parts,

$$\boldsymbol{B} = \langle \boldsymbol{B} \rangle + \boldsymbol{b}, \qquad \boldsymbol{U} = \langle \boldsymbol{U} \rangle + \boldsymbol{u}. \qquad (2)$$

The averaging procedure $\langle \cdot \rangle$ is assumed to obey the Reynolds averaging rules [2], which in the case of space or time averages generally implies the need for a wide scale separation (either spatial or temporal) between mean and fluctuating fields. Substitution for $\boldsymbol{B}$ and $\boldsymbol{U}$ from Eq. (2) into Eq. (1) yields [3]:

$$\frac{\partial \langle \boldsymbol{B} \rangle}{\partial t} = \nabla \times \langle \boldsymbol{u} \times \boldsymbol{b} \rangle + \frac{1}{Rm}\nabla^2 \langle \boldsymbol{B} \rangle, \qquad (3)$$

$$\frac{\partial \boldsymbol{b}}{\partial t} = \nabla \times (\boldsymbol{u} \times \langle \boldsymbol{B} \rangle) + \boldsymbol{G} + \frac{1}{Rm}\nabla^2 \boldsymbol{b}, \qquad (4)$$

where $\boldsymbol{G} = \nabla \times (\boldsymbol{u} \times \boldsymbol{b} - \langle \boldsymbol{u} \times \boldsymbol{b} \rangle)$ and where, for simplicity, we have assumed $\langle \boldsymbol{U} \rangle = 0$. The next step involves expressing the small-scale interactions in terms of the mean magnetic field. We proceed by assuming that there is no small-scale dynamo in the absence of a large-scale magnetic field [2] (but we note that for dynamos at high Rm one expects a small-scale dynamo to operate even in the absence of a mean magnetic field—see, e.g., [4]); thus $\boldsymbol{b}$ is linearly and homogeneously related to $\langle \boldsymbol{B} \rangle$ and we set

$$\mathcal{E} = \langle \boldsymbol{u} \times \boldsymbol{b} \rangle = \boldsymbol{\alpha} \cdot \langle \boldsymbol{B} \rangle + \boldsymbol{\beta} \cdot \nabla \langle \boldsymbol{B} \rangle + \ldots, \qquad (5)$$

where $\mathcal{E}$ is the electromotive force due to small-scale flow and field interactions, and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are tensors that depend on the statistical properties of the velocity field $\boldsymbol{u}$ and on Rm. Substitution of (5) into (3) yields an evolution equation for the mean magnetic field, which can then be solved in isolation. The coefficient $\boldsymbol{\alpha}$ may be decomposed into its symmetric and antisymmetric parts, $\alpha_{ij} = \alpha^s_{ij} - \epsilon_{ijk}\gamma_k$. The antisymmetric part results in an additional contribution ($\boldsymbol{\gamma}$) to the velocity in the mean-field equation [3]. The symmetric part (the "α effect") acts as a source term for the mean magnetic field, and leads to the fundamental difference between Eqs. (1) and (3). Since α^s_{ij} is the symmetric part of a pseudotensor, it is nonzero only if the velocity field lacks reflectional symmetry. It is the crucial ingredient for the generation of large-scale magnetic fields within mean-field electrodynamics and, accordingly, is the term on which we concentrate here.

The fundamental problem in mean-field electrodynamics is thus the determination of α_{ij} which, in general, may only be achieved through the solution of Eq. (4) for the fluctuating magnetic field $\boldsymbol{b}$. This is impossible analytically unless one resorts to some approximations. The most widely used is the first order smoothing approximation (FOSA) in which the nonlinear interactions between $\boldsymbol{u}$ and $\boldsymbol{b}$ (as manifested by $\boldsymbol{G}$) are neglected. This approximation can only be justified under two different but rather restrictive assumptions [2,3].

For $Rm \ll 1$, $\boldsymbol{G} \ll Rm^{-1}\nabla^2 \boldsymbol{b}$, and (4) becomes $\partial_t \boldsymbol{b} \approx \nabla \times (\boldsymbol{u} \times \langle \boldsymbol{B} \rangle) + Rm^{-1}\nabla^2 \boldsymbol{b}$. For stationary, homogeneous, isotropic turbulence (with $\alpha_{ij} = \alpha\delta_{ij}$) one obtains

$$\alpha = -\frac{1}{3} Rm \iint \frac{k^2 F(k, \omega)}{Rm^2 \omega^2 + k^4} dk d\omega, \tag{6}$$

where $F(k, \omega)$ is the helicity spectrum function of the velocity field.

Alternatively, in the "short-sudden" approximation—i.e., if the velocity correlation time τ_c is far shorter than its turnover time τ_0—then $\boldsymbol{G} \ll \partial_t \boldsymbol{b}$. This approximation is often used in conjunction with the high conductivity limit $Rm \gg 1$; Eq. (4) then becomes $\partial_t \boldsymbol{b} \approx \nabla \times (\boldsymbol{u} \times \langle \boldsymbol{B} \rangle)$. Analytic progress can be made for stationary, homogeneous, isotropic turbulence, leading to

$$\alpha = -\frac{\tau_c}{3} \langle \boldsymbol{u} \cdot \nabla \times \boldsymbol{u} \rangle, \tag{7}$$

where $\langle \boldsymbol{u} \cdot \nabla \times \boldsymbol{u} \rangle$ is the helicity of the flow. More generally, the expressions for $\boldsymbol{\alpha}$ are more involved; nonetheless, in the analogues of Eqs. (6) and (7) $\boldsymbol{\alpha}$ is related to Rm and τ_c respectively in a linear fashion, and in both cases to the helicity of the flow.

For both of these approximations, therefore, α is directly related to the helicity of the flow, but takes the opposite sign. Moreover, the simplest measure of the lack of reflectional symmetry in a flow is given by the helicity. These considerations, together with Parker's picture of the generation of mean field via the ensemble action of a series of cyclonic events [5], have led to the widespread belief that the α effect generated by turbulent motions is related in a simple way to the helicity. We stress here that expressions (6) and (7) are derived under two assumptions, neither of which is likely to be valid in an astrophysical context. Clearly $Rm \ll 1$ is inappropriate but also, for conventional turbulence, $\tau_c \sim \tau_0$, which is outside the range of applicability of the short-sudden approximation. Moreover, at high Rm, numerical calculations indicate that $|\mathbf{b}| \gg |\langle \mathbf{B} \rangle|$ [6], so that the term balance leading to (7) cannot hold. Given that in astrophysical modelling, α is often parametrized as having a straightforward relationship to the helicity (and indeed expressions such as Eqs. (6) and (7) are often quoted as justification for such a parametrization) it is important to investigate whether such a direct relationship holds in general. In this Letter we therefore investigate the dependence of the α effect on Rm and τ_c for a family of helical flows.

Astrophysical flows are typically highly turbulent and three-dimensional with huge values of the fluid and magnetic Reynolds numbers. Direct numerical simulation of such flows is impossible and thus any astrophysical simulation necessarily requires simplification. One approach is to simulate fully three-dimensional flows, in which case one is limited to moderate [$O(100)$] Reynolds numbers. Alternatively one may simplify the structure of the flow in order to achieve high [$O(10^5)$] Reynolds numbers, which is the approach adopted here. We solve Eq. (4) numerically and calculate $\boldsymbol{\alpha}$ for velocities of the form

$$\boldsymbol{u} = (\partial_y \psi, -\partial_x \psi, -\psi), \tag{8}$$

where the two-dimensional stream function $\psi(x, y; t)$ is chosen to be 2π periodic in space. Notice that although the velocity depends on only the x and y coordinates, all three components are nonzero, and so it is capable of generating magnetic fields via dynamo action. Flows of the form (8) are homogeneous but highly anisotropic. The two-dimensional nature of the velocity field is advantageous for two reasons. First, it allows a two-dimensional calculation of $\boldsymbol{\alpha}$; consequently, as discussed above, extremely large values of Rm may be considered. Second, if the field is also assumed to be independent of the z coordinate then no small-scale dynamo action is possible [7]; thus, in the absence of an imposed mean field, the magnetic field will decay. Hence the small-scale field is guaranteed to be linearly and homogeneously related to the mean magnetic field, as assumed in deriving Eq. (5). The *small*-scale dynamo properties of flows (8), for different choices of $\psi(x, y; t)$, have been extensively studied for magnetic fields of the form $\boldsymbol{b} = \tilde{\boldsymbol{b}}(x, y; t)e^{ikz}$ [8–10]. In contrast, much less is known about the *large*-scale dynamo properties of such flows, although $\boldsymbol{\alpha}$ has been calculated for a subset of steady flows, as discussed below.

Here we determine the tensor $\boldsymbol{\alpha}$ by solving Eq. (4) with $\boldsymbol{u}$ given by Eq. (8) and with the imposition of a mean field $\boldsymbol{B}_0 = (B_0, 0, 0)$ in order to calculate self-consistently the emf $\mathcal{E} = \langle \boldsymbol{u} \times \boldsymbol{b} \rangle$. We utilize a two-dimensional pseudospectral discretization in space together with a second-order Runge-Kutta timestepping scheme. The initial condition $\boldsymbol{b}(x, y, t) = 0$ is chosen since, together with the choice of periodic boundary conditions, this ensures that there is no other contribution to the mean field than the imposed $\boldsymbol{B}_0$. This two-dimensional ansatz is formally the limit of a three-dimensional dynamo calculation with k tending to zero [9,11]. With a uniform mean field, Eq. (5) simply yields $\boldsymbol{E} = \boldsymbol{\alpha} \cdot \boldsymbol{B}_0$. Moreover, since the flows are independent of z, the α effect is anisotropic and occurs in the xy plane; we therefore concentrate on the 2×2 part of $\boldsymbol{\alpha}$ that relates "horizontal" quantities.

We focus on two choices of ψ. Initially

$$\psi(x, y, t) = \sqrt{\frac{3}{2}}(\cos(x + \epsilon \cos t) + \sin(y + \epsilon \sin t)), \tag{9}$$

with ϵ a parameter that is allowed to vary. For $\epsilon = 0$ the flow is steady and hence integrable; any dynamo that results must therefore switch off as $Rm \to \infty$ [12,13]. The α effect for two-dimensional steady flows has also been calculated [9,14] and, in agreement with asymptotic studies [15,16], tends to zero with increasing Rm. For $\epsilon \neq 0$ the flows are time dependent and display Lagrangian chaos, with the parameter ϵ controlling the size of the chaotic regions [17]. The small-scale dynamo properties of the flow with $\epsilon = 1.0$ are well documented [4,10]. In particular, these flows are known to be good candidates for fast dynamo action, with the growth rate approaching a positive asymptotic value as $Rm \to \infty$.

Astrophysical dynamos are however characterized by the generation of *large*-scale magnetic fields by turbulent flows at high Rm. It is therefore vital to understand the behavior of $\boldsymbol{\alpha}$ in this regime. Specifically, one must therefore consider chaotic flows and, to this end, we determine $\boldsymbol{\alpha}$, as Rm is increased to large values, for flows (9) with $\epsilon \neq 0$. These are invariant under a 90° rotation about the z axis with appropriate shifts in space and time; $\boldsymbol{\alpha}$, as a mean quantity, is similarly invariant and therefore takes the form $\alpha_{ij} = \alpha\delta_{ij} - \epsilon_{ij3}\gamma$. Thus $\alpha = \mathcal{E}_x B_0^{-1}$ and $\gamma = \mathcal{E}_y B_0^{-1}$.

Figure 1 shows a typical time series for $\langle \boldsymbol{u} \times \boldsymbol{b} \rangle_x$, where, from now on, $\langle \cdot \rangle$ denotes an average over the xy plane. The final state consists of oscillations about a nonzero mean value $\mathcal{E}_x$—which is the value used to calculate α. Similarly $\langle \boldsymbol{u} \times \boldsymbol{b} \rangle_y$ is used to calculate γ. Figure 2 summarizes a series of calculations of α and γ for a range of values of Rm and ϵ. Two main conclusions can be drawn. First, α and γ depend sensitively on Rm. For the range of Rm that we have investigated (up to $Rm = 2 \times 10^5$ for $\epsilon = 1.0$) we have been unable to determine an asymptotic, high Rm limit for α and γ. It remains an open question as to whether such a limit does indeed exist and, if it does, whether the value of Rm needed to attain this limit is related to the degree of chaos in the flow. It is worth noting that, by comparison, the small-scale dynamo growth rate for the flow with $\epsilon = 1.0$ reaches its asymptotic limit by $Rm \sim 100$. Moreover, the values of the coefficients at high Rm do not appear to be related to their values at lower Rm, where the FOSA is appropriate. Second, and more striking, is the fact that α and γ *change sign* as Rm is increased. We stress again here that the helicity of the flow is the same for all values of Rm. The curves corresponding to the different values of ϵ diverge significantly as soon as $Rm \gtrsim 1$. For $Rm \ll 1$, the flows behave according to the FOSA. For higher Rm, however, α and γ vary significantly with ϵ, and hence with the chaotic properties of the flow. It seems therefore that it is not possible in this case to relate the α effect solely to mean Eulerian properties of the flows (such as helicity): its dependence on correlations between the fluctuating parts of the velocity and magnetic fields through $\boldsymbol{G}$ in (4) must be taken into account.

It may be argued that the strong dependence of the transport coefficients α and γ on Rm and ϵ are the result of considering time-periodic flows, i.e., flows with an infinite correlation time. As noted above, the FOSA in the high conductivity limit requires $\tau_c \ll 1$, so it is of interest to see how far this approximation can be extended to flows with correlation times of order unity or longer. Therefore we now consider a random version of the flow (9) in which the stream function is defined as

$$\psi = \sqrt{\frac{3}{2}}(\cos\{x + \epsilon\cos[t + \phi(t)]\} + \sin\{x + \epsilon\sin[t + \phi(t)]\}), \tag{10}$$

where the phase $\phi(t)$ varies on a time scale τ_c. The variation takes the form of long intervals in which ϕ remains constant, at a value randomly selected from a uniform distribution with $0 < \phi \leq 2\pi$, interspersed with intervals where ϕ varies rapidly between these constant values. Figure 3 shows typical time series of $\langle \mathbf{u} \times \mathbf{b} \rangle_x$ for two different correlation times. Since the time traces are now random, in order to obtain meaningful averages α must be evaluated by averaging the emf $\langle \boldsymbol{u} \times \boldsymbol{b} \rangle_x$ over a time long compared with the diffusive time scale.

Figure 4 shows α and γ as a function of Rm for $\epsilon = 0.75$ and for $\tau_c = 1.57$, 3.75, and 37.5. For comparison, the other relevant time scales are the turnover time $\tau_0 = Lu_{\rm rms}^{-1} \sim 3.63$, the time-periodicity 2π, and the diffusive time scale $\tau_\eta \sim \mathcal{O}(Rm)$. For low Rm, when diffusion dominates, the curves again superpose. For short correlation times, α and γ vary little as Rm is increased and seem to become independent of Rm as soon as $Rm \sim \mathcal{O}(1)$. For longer correlation times, α exhibits a strong Rm depen-

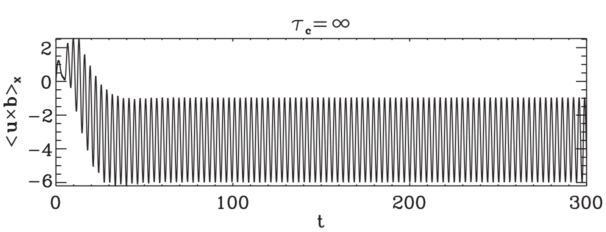

FIG. 1. Typical temporal evolution of $\langle \boldsymbol{u} \times \boldsymbol{b} \rangle_x$ for flow (9) with $\epsilon = 0.75$, $Rm = 64$, $B_0 = 1$.

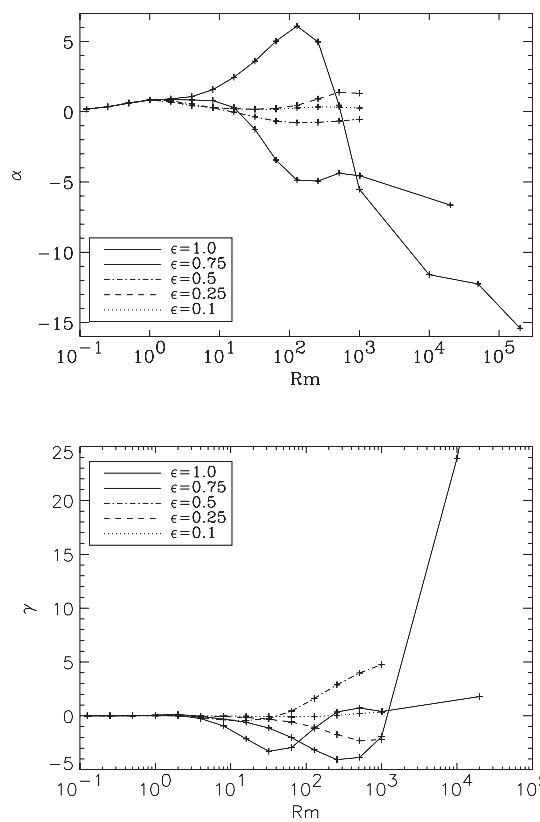

FIG. 2. Dependence of α and γ on Rm for flow (9) for different values of ϵ.

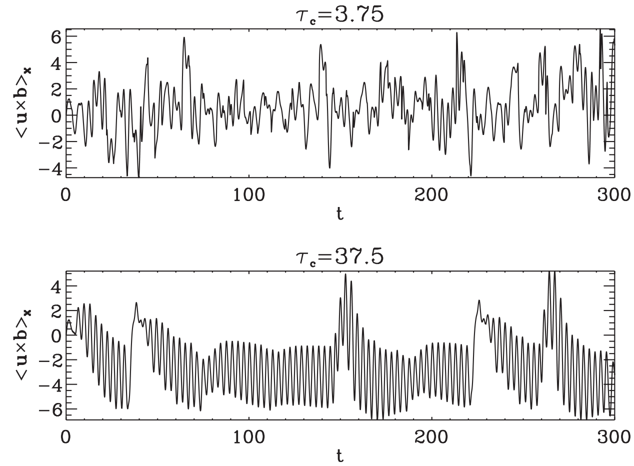

FIG. 3. $\langle \mathbf{u} \times \mathbf{b} \rangle_x$ as a function of time for flow (10) with $Rm = 64$, $\epsilon = 0.75$, and $\tau_c = 3.75$ (top), $\tau_c = 37.5$ (bottom).

dence with a sign change, as in the previous example, but seems to approach an asymptotic limit as Rm increases. We conjecture that, for any finite correlation time, α attains an Rm-independent value α^* for $Rm > Rm^*$, with Rm^* an increasing function of τ_c. If τ_c is infinite, α and γ may never settle down.

An important question is what determines α^* (and γ^*). In the limiting case of extremely small τ_c, α^* will be independent of ϵ but linearly related to τ_c and to the helicity (although here α may not go to zero as $\tau_c \to 0$ owing to the presence of an underlying systematic flow). Further simulations demonstrate that, even for short correlation times, α^* and γ^* are still sensitive to the value of ϵ and hence to the chaotic properties of the flow. Hence, for finite correlation times, even though α and γ become Rm independent, the structure of the flow still determines their asymptotic values; indeed, even the sign of α cannot be determined *a priori*.

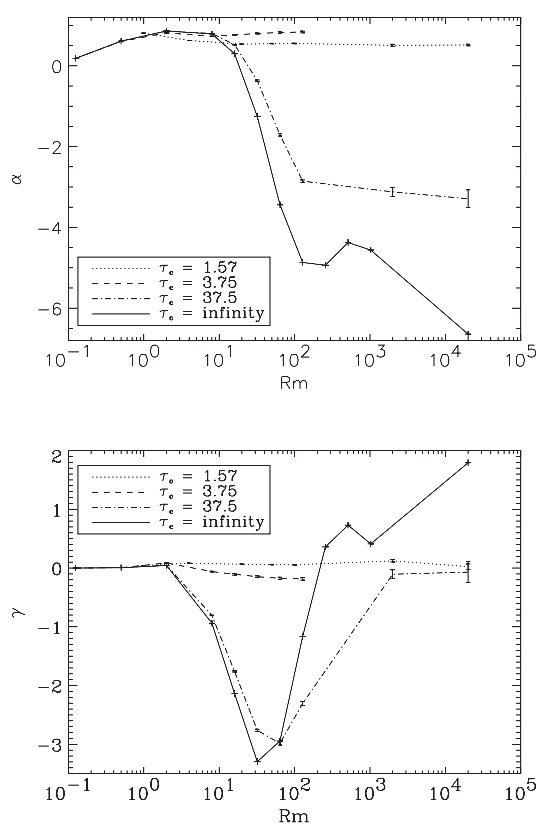

FIG. 4. Dependence of α and γ on Rm for flow (10) for different values of τ_c and $\epsilon = 0.75$.

The present work emphasizes that the α effect in the high conductivity limit, even in the kinematic regime, remains a delicate issue. What is clear, however, is that simple prescriptions such as those given by expressions (6) and (7) are incorrect outside their extremely limited range of applicability. In a different context, an initial departure from FOSA has also been noted for a geodynamo simulation at moderate Rm [18]. Models that rely on such simple parametrizations are therefore likely to be subject to significant and undetermined errors. It is therefore essential to gain a better understanding of the basic physical processes involved in field transport instead of relying too heavily on *ad hoc* parametrizations of mean-field models.

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