Resonant Excitation of Rossby Waves in the Equatorial Waveguide and their Nonlinear Evolution

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Nonlinear interactions between the baroclinic Rossby waves trapped in the equatorial waveguide and the barotropic Rossby waves freely propagating across the equator are studied within the two-layer model of the atmosphere, or the ocean. It is shown that a barotropic wave can resonantly excite a pair of baroclinic waves with amplitudes much greater than its proper amplitude. The envelopes of the baroclinic waves obey Ginzburg-Landau-type equations and exhibit nonlinear saturation and formation of characteristic "domain-wall" and "dark-soliton" defects.

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Because of the change of sign of the vertical component of the Earth's angular velocity at the equator this region in the atmosphere and oceans is dynamically special. There exists a whole family of specific waves localized in the so-called equatorial waveguide [1]. These equatorial waves, and, in particular, the Rossby waves, are known to play an important role in the dynamical processes both in the ocean and in the atmosphere which determine Earth's climate, such as the El Niño phenomenon [2,3], or tropospheric Madden-Julian oscillation, [4]. Together with the equatorial waveguide modes, nonlocalized planetary Rossby waves freely propagating across the equator also exist. The equator, thus, represents a "semi-transparent" waveguide.

Below we study nonlinear interactions of the localized and nonlocalized Rossby modes. Our analysis demonstrates a resonant excitation of the waveguide waves by incoming nonlocalized waves with subsequent nonlinear saturation at the level greatly exceeding the amplitude of the incoming wave. In their turn, nonlinear interactions among the waveguide modes modify the incoming wave. This mechanism of energy exchange between midlatitudes and the equatorial region is new. The envelopes of the waveguide modes obey a generalized Ginzburg-Landau (GL) equation, or a pair of coupled GL-type equations with coefficients of special structure. We believe these results are not limited to equator and are generic for semi-transparent waveguides of various origins.

We use the two-layer model of equatorial dynamics. In terms of the nondimensional barotropic stream function ψ, baroclinic velocity $\mathbf{u} = (u, v)$, and the depth of the upper layer h [for details see [5]], the equations of the model become:

$$\nabla^2\psi_t + \psi_x = \epsilon[-J(\psi, \nabla^2\psi) - s(\partial_{xx} - \partial_{yy})[(1+\epsilon q h)(uv)] + s\partial_{xy}[(1+\epsilon q h)(u^2 - v^2)]] \quad (1)$$

$$\mathbf{u}_t + \nabla h + y\hat{\mathbf{z}}\times\mathbf{u} = \epsilon[-J(\psi, \mathbf{u}) + \mathbf{u}\cdot\nabla(\hat{\mathbf{z}}\times\nabla\psi) - q\mathbf{u}\cdot\nabla\mathbf{u} + \epsilon s(2h\mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{u}\mathbf{u}\cdot\nabla h)], \quad (2)$$

$$h_t + \nabla\cdot\mathbf{u} = \epsilon[-J(\psi, h) - q\nabla\cdot(\mathbf{u}h) + \epsilon s\nabla\cdot(h^2\mathbf{u})]. \quad (3)$$

Here and below the subscripts denote the corresponding partial derivatives, ϵ is the nonlinearity parameter, and J denotes the Jacobian. The model is of frequent use in studies of linear [6] and nonlinear [7] equatorial dynamics and may be obtained from the full equations for the rotating continuously stratified fluid by a number of methods [6,8]. The values of the parameters $s \neq 0$ and q depend on the method of derivation.

In the linear approximation the model describes, among others, the barotropic Rossby waves which can propagate at any angle with respect to the equator:

$$\psi_0 = A_\psi e^{i(\theta+ly)} + \text{c.c.}; \qquad \theta = kx - \sigma t; \quad (4)$$

with the dispersion relation

$$\sigma = -k/(k^2 + l^2), \quad (5)$$

and the trapped baroclinic Rossby waves

$$(\mathbf{u}_0, h_0) = (\mathbf{U}_m, H_m)Ae^{i\theta_m} + \text{c.c.}; \qquad \theta_m = \hat{k}x - \sigma_m t \quad (6)$$

with the dispersion relation

$$\sigma_m^3 - (\hat{k}^2 + 2m + 1)\sigma_m - \hat{k} = 0; \qquad m = 1, 2, \ldots. \quad (7)$$

Here m is the meridional wave number and the functions $\mathbf{U}_m = (U_m, V_m)$, H_m are strongly localized near the equator $y = 0$ and have the form $P(y)e^{-(y^2/2)}$, where $P(y)$ is a normalized polynomial of degree m or $m + 1$ [1].

We study nonlinear interactions of the waves (4) and (6), at small nonlinearities $\epsilon \ll 1$, by using the asymptotic expansions of all fields in ϵ and eliminating the resonances. As usual, the slow time dependence on $T = \epsilon t$ of the wave amplitudes A_ψ, A is introduced:

$$(\psi, \mathbf{u}, h) = (\psi_0, \mathbf{u}_0, h_0)(x, y, t, T) + \epsilon(\psi_1, \mathbf{u}_1, h_1)(x, y, t, T) + \cdots \quad (8)$$

The dispersion relations (5) and (7) allow for the triadic synchronism conditions:

$$\hat{k}_1 \pm \hat{k}_2 = k; \sigma_{m_1} \pm \sigma_{m_2} = \sigma. \quad (9)$$

The key observation is that the interactions among baroclinic Rossby waves do not generate resonances in the right-hand side of (1) and the influence of the baroclinic waves on the barotropic one may be neglected as long as the amplitudes of the baroclinic waves $A_{1,2}$ are not too large. Hence, the resulting equation for triadic interactions may be reduced to:

$$A_{i_{TT}} = K|A_\psi|^2 A_i, \, i = 1, 2. \quad (10)$$

Here K is a constant real coefficient (see below), and A_ψ does not change in time. K is of the order one and positive in the upper case in (9) and negative in the lower case. Hence, the barotropic Rossby mode can excite the exponentially growing baroclinic Rossby waves with frequencies lower than its proper one. In what follows we will concentrate on this most interesting case.

On longer times a secondary barotropic wave $\epsilon\psi_1$ in (8) generated by self-interaction of the baroclinic modes becomes comparable to ψ_0 and its interaction with baroclinic modes arrests their growth. To study the stage of nonlinear saturation the asymptotic expansion in ϵ should be rearranged as follows:

$$\begin{aligned}\psi &= \psi_0(x, y, t, T_1, T_2, \ldots) \\ &\quad + \epsilon^{1/2}\psi_1(x, y, t, T_1, T_2, \ldots) + \cdots, \\ (\mathbf{u}, h) &= \epsilon^{-1/2}(\mathbf{u}_0, h_0)(x, y, t, T_1, T_2, \ldots) \\ &\quad + (\mathbf{u}_1, h_1)(x, y, t, T_1, T_2, \ldots) + \cdots,\end{aligned} \quad (11)$$

with slow times $T_n = \epsilon^{n/2}$. In this expansion ψ_0 contains both primary and secondary barotropic waves. We limit ourselves by giving the results of the asymptotic analysis which is rather straightforward; cf. [9] for a similar study.

We start with an interesting particular case where the baroclinic modes forming a triad with the barotropic one are identical:

$$2\hat{k} = k, \qquad 2\sigma_m = \sigma. \quad (12)$$

Analysis of the dispersion relations (5) and (7) shows that this is possible for small enough k. A standard procedure of elimination of resonances leads to the following equation for the amplitude of the baroclinic wave (the bar means complex conjugation):

$$A_T + \alpha\bar{A} + \beta|A|^2 A = 0. \quad (13)$$

The coefficients are of order one and are determined by the structure of the wave once the value of k is fixed:

$$\begin{aligned}\alpha &= A_\psi \frac{\int_{-\infty}^{\infty} dy F(y) \sin|l|y}{\int_{-\infty}^{\infty} dy(\mathbf{U}_m^2 + H_m^2)}, \\ \operatorname{Re}\beta &= \frac{s}{2|l|\sigma} \frac{(\int_{-\infty}^{\infty} dy F(y) \sin|l|y)^2}{\int_{-\infty}^{\infty} dy(\mathbf{U}_m^2 + H_m^2)} > 0,\end{aligned} \quad (14)$$

with

$$F(y) = [V_m(y)U_m(y)]'' - 2k[\mathbf{U}_m^2(y)]' + 4k^2 V_m(y)U_m(y). \quad (15)$$

(We do not give the expression of Im β which has a similar structure but is rather cumbersome.) For small nonlinearities this equation reproduces (10) with positive $K = \frac{|\alpha|^2}{|A_\psi|^2}$.

The full equation (13) has two stationary solutions

$$A_\pm^2 = -\frac{\alpha}{\beta} \quad (16)$$

which are stable for $\operatorname{Re}\beta > 0$, and attractive, at least for the initial values of A in the vicinity of zero. Thus nonlinear saturation of a growing baroclinic wave always takes place. $A_\pm$ are of the order unity; therefore, the saturated baroclinic amplitudes greatly exceed (by the factor $\epsilon^{-1/2}$) the amplitude of the initial barotropic wave; cf. (11).

Up to now we considered monochromatic spatially uniform waves. If spatial modulation of the excited wave is to be studied, a hierarchy of slow modulation space scales $X_1 = \epsilon^{1/2}x, X_2 = \epsilon x, \ldots$ should be introduced. By working in a reference frame moving with the group velocity of the baroclinic wave $c_g(\hat{k})$ a space-time counterpart of (13) is obtained after proper renormalizations:

$$A_{T_2} - ic_g'(\hat{k})A_{X_1X_1} + \alpha\bar{A} + \beta|A|^2 A = 0. \quad (17)$$

This is an equation of the GL type. It falls into the class of so-called resonantly forced GL equations known in the literature for various physical situations where parametric excitation of waves takes place; cf. [10] and references therein. However, the mechanism of excitation and saturation is different from the standard parametric excitation. First, the barotropic wave does not act as pure external forcing, being changed by the secondary barotropic wave. Second, the resonantly driven GL equation commonly studied in literature [10–12] contains the term μA with $\operatorname{Re}\mu \neq 0$. In our case this term is absent [13]. The case closest to ours arises in the theory of the so-called edge waves on the beaches [9], see below.

The stationary solutions (16) are still solutions of (17) and it is easy to see that they are stable for $\operatorname{Re}\beta > 0$, which is the case. The fact that there are two different stationary states makes one think of nontrivial solutions of the domain-wall type, as it is the case for similar GL equations [10]. A typical result of the numerical simulations of the evolution of an initially localized A is shown on the Figs. 1 and 2. A characteristic Bloch-type [cf. [11]] domain-wall structures appear indeed, forming a bound state (a so-called bubble, or "dark soliton"), of the type studied,

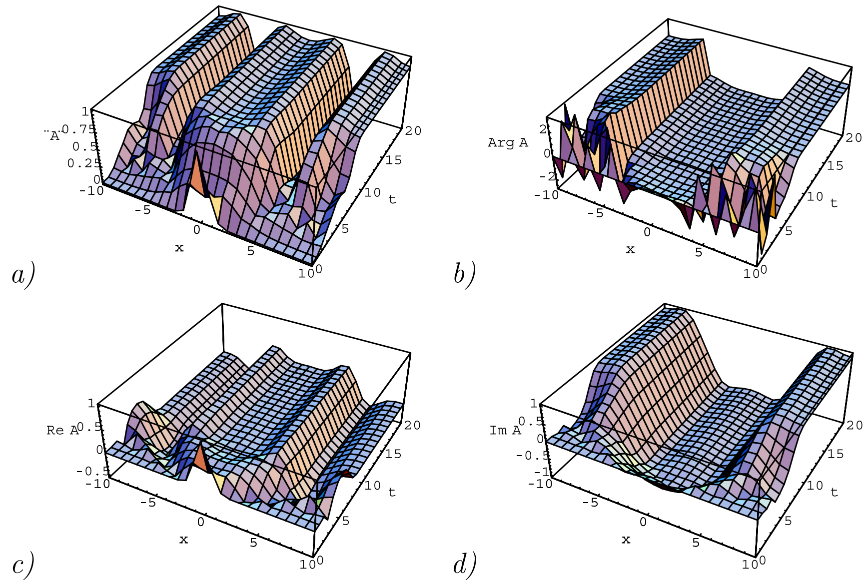

FIG. 1 (color online). Space-time evolution of a localized wave packet. (a) The wave amplitude $|A|$, (b) the wave-phase arg$A$, (c) the real part Re A, (d) the imaginary part Im A. The calculation was performed with a standard MATHEMATICA partial differential equation solver with spatially periodic boundary conditions, starting from the Gaussian initial distribution of Re A and zero Im A. The values of parameters are $\alpha = e^{i\pi/4}$, $\beta = 1 + 0.5i$.

e.g., in [14] although, as said, the structure of equations is different.

In the general case, a barotropic Rossby wave excites a pair of different baroclinic waves. Without taking into account the spatial modulation, the equations for the amplitudes of the baroclinic waves are:

$$\begin{aligned} A_{1_T} + \alpha_1\bar{A}_2 + \beta_1|A_1|^2A_1 + \gamma_1|A_2|^2A_1 = 0, \\ A_{2_T} + \alpha_2\bar{A}_1 + \beta_2|A_2|^2A_2 + \gamma_2|A_1|^2A_2 = 0, \end{aligned} \qquad (18)$$

where Re $\beta_{1,2} \geq 0$. It may be easily shown that this system, in general, does not admit stationary solutions. Instead a harmonically oscillating solution of the form:

$$A_1 = \hat{A}_1 e^{i\omega T}, \qquad A_2 = \hat{A}_2 e^{-i\omega T}, \qquad (19)$$

exists, where $\hat{A}_{1,2}$ are complex constants and ω is a real frequency. Like (16) this solution is attractive, as direct numerical tests show.

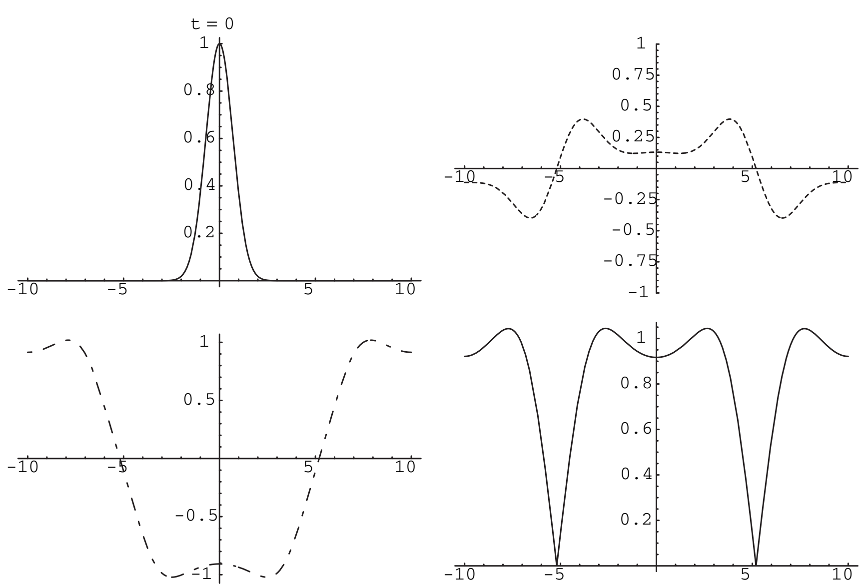

FIG. 2. Initial (top left), and final profiles of the real (dashed line) and imaginary (dash-dotted line) parts of A, and its amplitude (solid line), in the calculation of Fig. 1.

Introduction of the slow spatial modulation gives in the leading order the following system of partial differential equation:

$$A_{1_T} + c_g(\hat{k}_1)A_{1_X} + \alpha_1 \bar{A}_2 + \beta_1|A_1|^2 A_1 + \gamma_1 |A_2|^2 A_1 = 0,$$
$$A_{2_T} + c_g(\hat{k}_2)A_{2_X} + \alpha_2 \bar{A}_1 + \beta_2|A_2|^2 A_2 + \gamma_2 |A_1|^2 A_2 = 0. \quad (20)$$

Because of the presence of two different group velocities of the baroclinic waves, it is impossible to get rid of the terms with the first spatial derivative, as it was done in the case of a single baroclinic wave above. The terms with second spatial derivatives, thus, appear as the next order corrections. The system (20) is a hyperbolic system with straight characteristics $\frac{dX}{dT} = c_g(\hat{k}_{1,2})$. For localized initial distributions of $A_{1,2}$ the angle between the characteristics determines the zone of influence of the initial conditions. The behavior of the amplitudes within this angle is similar to that predicted by (18). Resemblance of the system (20) to the one describing dynamics of counterpropagating waves in the Faraday effect [15] is to be emphasized, with an important difference of absence of linear in $A_{1,2}$ terms, and different structure of the coefficients.

The physical picture arising from the presented results is as follows. In the linear approximation the equatorial waveguide is transparent for the barotropic Rossby waves. Because of nonlinear effects, the barotropic wave resonantly excites (for instance from the preexisting noise) a pair of the baroclinic waveguide modes with exponentially growing amplitudes. In their turn, the interacting baroclinic waves give rise to an exponentially growing secondary barotropic mode. This mode has the form of reflected and transmitted waves spreading with time out of the equator. Its interaction with the baroclinic modes arrests the growth of these latter. The amplitudes of the excited baroclinic waves are rapidly saturated but exhibit characteristic domain-wall-like phase defects and ''dark-soliton'' structures. The equator, thus, represents a semitransparent waveguide where the waveguide modes are resonantly excited by nonlocalized external modes. Inversely, the nonlocalized modes are modified by waveguide modes. We believe that this situation is generic. Although one can imagine semitransparent waveguides of various natures, the only example treated in literature we are aware of is the beach edge waves. These waves trapped near the shore may be resonantly excited by the waves coming onshore from the open ocean [9,16]. Although the scales, the physics of the system, and the dispersion properties of the waves are very different from the equatorial waves, the resulting modulation equations are close. However, they were not, as to our knowledge, exhaustively studied. The results of the present Letter apply, at least qualitatively, to the edge waves, too.

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