Chaos Control using Notch Filter Feedback

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A method for stabilizing periodic orbits and steady states of chaotic systems is presented using specifically filtered feedback signals. The efficiency of this control technique is illustrated with simulations (Rössler system, laser model) and a successful experimental application for stabilizing intensity fluctuations of an intracavity frequency-doubled Nd:YAG laser.

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During the past 15 years many methods have been devised for controlling chaotic systems by stabilizing some unstable periodic orbits (UPOs) or fixed points which are embedded in the chaotic attractor [1–4]. A very efficient class of control schemes that can be implemented relatively easily in experiments are delayed feedback methods first applied to chaotic systems by Pyragas [5] and later analyzed and extended by different authors [6–13]. All delayed feedback methods provide linear control schemes that can be described using transfer functions $T(f)$ in frequency space [7,13]. As a typical example Fig. 1(c) shows the absolute value of the transfer function of Pyragas' time delay autosynchronization (TDAS) control [5] that is given by $T(f) = \exp(-i2\pi f\tau) - 1$. If the delay time τ equals the period T of some specific UPO then all notches of the transfer function coincide with the spectral lines (harmonics) of the UPO. Therefore, the UPO is not affected by the feedback (noninvasive control) in contrast to the "remaining" chaotic dynamics which is damped out due to the feedback. This mechanism is illustrated in Fig. 1 using the chaotic Rössler system

$$\dot{x} = -y - z \qquad \dot{y} = x + 0.2y - u$$
$$\dot{z} = 0.2 + (x - 5.7)z. \tag{1}$$

Figure 1(a) shows the power spectrum and Fig. 1(b) the time series if no control is applied [$u(t) = 0$]. The spectrum [Fig. 1(a)] possesses some dominant peaks corresponding to UPOs embedded in the chaotic attractor that can be stabilized using Pyragas' TDAS control

$$u(t) = k[y(t) - y(t - \tau)] \tag{2}$$

with delay time $\tau = T$ where T denotes the period of the UPO to be stabilized. As an example, Fig. 1(d) shows a stabilized periodic oscillation of the Rössler system (1) obtained with $k = 0.2$ and $\tau = 5.85$.

Such a stabilization of a periodic oscillation can also be achieved by replacing the delay line in Eq. (2) by a (linear) notch filter as shown in Figs. 1(e) and 1(f). The notch filter used here is given by a Wien-bridge [14]

$$\dot{U}_{C_1} = \frac{1}{RC}\left(U_{\text{in}} + \frac{2Q-1}{Q}U_{C_2} - U_{C_1}\right)$$
$$\dot{U}_{C_2} = \frac{1}{RC}\left(U_{\text{in}} + \frac{Q-1}{Q}U_{C_2} - U_{C_1}\right) \tag{3}$$
$$U_{\text{out}} = \frac{1}{Q}U_{C_2},$$

where U_{in} and U_{out} denote the input and the output signal of the notch filter, respectively. U_{C_1} and U_{C_2} are capacitor voltages, Q is the quality of the filter, and RC determines the resonance frequency $f_r = (2\pi RC)^{-1}$ of the notch filter. Using this filter the control signal applied to the Rössler system (1) is given by

$$u(t) = k[U_{\text{in}}(t) - U_{\text{out}}(t)], \tag{4}$$

where U_{in} is chosen here to be the y component of Eq. (1). Figure 1(e) shows the transfer function of this control

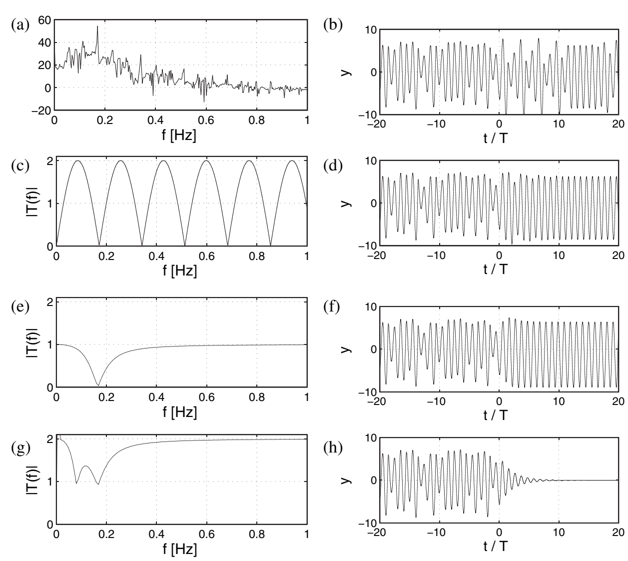

FIG. 1. Different control methods applied to the chaotic Rössler system (1). Spectrum (a) and chaotic time series (b) of the free running system ($u = 0$). Transfer function $|T(f)|$ and time series of the controlled system (feedback switched on at $t = 0$) for: (c), (d) TDAS control (2), (e), (f) single notch filter control (4), and (g), (h) two notch filters control (5).

scheme (4) for $Q = 2.3$, $R = 960\ \Omega$, $C = 1$ mF. Although strictly speaking this is (in contrast to TDAS) an invasive control method [$u(t)$ does not vanish] the resulting periodic oscillation is very close to the UPO stabilized with TDAS in Fig. 1(d).

Stabilization of steady states (fixed points) can be achieved using additional notch filters (in parallel) which suppress any spectral components of the dynamical system that would be excited without control. The case of two notch filters is illustrated in Fig. 1(g) showing the transfer function of the corresponding feedback loop

$$u(t) = k_1[U_{\mathrm{in}} - U^1_{\mathrm{out}}] + k_2[U_{\mathrm{in}} - U^2_{\mathrm{out}}] \tag{5}$$

for $k_1 = k_2 = 0.2$, $Q_1 = Q_2 = 1.3$, $C_1 = C_2 = 1$ mF, $R_1 = 520\ \Omega$, and $R_2 = 1950\ \Omega$. Figure 1(h) shows the stabilization of a steady state using as filter input $U_{\mathrm{in}} = y$. This steady state, however, slightly deviates from the fixed point $(x_0, y_0, z_0) = (-0.00702, -0.0351, 0.0351)$ of the Rössler system (1) because the control signal $u(t)$ does not vanish but converges to a fixed value shifting the fixed point.

A noninvasive control scheme for stabilizing an unstable fixed point of the free running system is obtained by high-pass filtering the input signal U_{in} of the notch filter. In this case the control signal $u(t)$ vanishes once the fixed point is reached and does not produce an offset of the steady state.

In comparison to the transfer function of multiple delay feedback control using several delay times [13] notch filter feedback possesses additional quality parameters that may be adjusted to the given control task.

The performance of notch filter control depends crucially on the resonance frequencies. This is illustrated in Fig. 2 for the case of two notch filters applied to stabilize the fixed point of the Rössler system (1) using the ac component of y as input signal. A linear stability analysis was performed for the controlled system and the eigenvalue λ with the largest real part $\mathrm{Re}(\lambda)$ was determined. Fixed point stabilization is successful if this real part is negative. This is visualized in Fig. 2 by means of the stability function $\max[0, -\mathrm{Re}(\lambda)]$ which vanishes at parameter values (notch filter frequencies) where control fails and whose positive values are a measure of robustness of the achieved stabilization. If the resonance frequencies $f_i = (2\pi R_i C)^{-1}$ of the notch filters are chosen close to the main frequency ≈ 0.17 Hz of the Rössler system the fixed point remains unstable because this crucial spectral component is not fed back. As a result, the stability function equals zero as can be seen as a white spot in the center of Fig. 2. On the other hand, high stability can be achieved if the notch filter frequencies are chosen differently from each other.

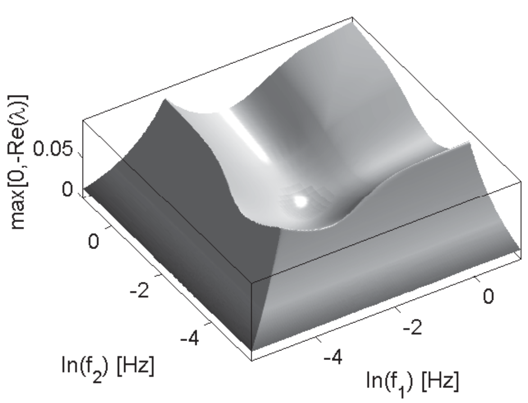

FIG. 2. Stability function $\max[0, -\mathrm{Re}(\lambda)]$ vs resonance frequencies f_1 and f_2 of the applied notch filters ($k_1 = k_2 = 1.5$, $Q_1 = Q_2 = 0.5$) used for stabilizing the fixed point of the Rössler system (1). $\mathrm{Re}(\lambda)$ denotes the real part of the largest eigenvalue. The system is stable if all eigenvalues possess negative real part, i.e., if $\max[0, -\mathrm{Re}(\lambda)]$ is positive.

Feedback control using notch filters can easily be implemented in analog hardware. This feature makes it, in particular, interesting for fast (chaotic) dynamics. An example where high frequency chaotic oscillations occur are compact intracavity frequency-doubled Nd:YAG lasers, exhibiting intensity fluctuations due to multimode operation [15,16]. This phenomenon is known in literature as *green problem* and is still a topic of current research due to its relevance for technical applications where intensity fluctuations have to be avoided. Most attempts to tame this chaotic instability are based on optical modifications [17–21] or feedback techniques [22–28]. For higher pump currents [type-II-chaos [29,30]] we succeeded in experimental stabilization of the laser's steady state using multiple delay feedback [12]. These results were also confirmed by numerical simulations [31] based on a physical model for frequency-doubled lasers introduced by Pyragas *et al.* [27] and a feedback signal consisting of differently delayed infrared signals. In the following we shall demonstrate that stabilization of the laser can also be achieved using notch filters in the feedback loop instead of delay lines. The model we used for simulating an intracavity frequency-doubled solid state laser [27] consists of dynamical equations for the intensities I_j of the optical modes

$$\frac{dI_j}{d\vartheta} = \left\{G_j - \frac{\epsilon}{\alpha\eta}\left[g(I_j - I_j^0) + 2\sum_{i\neq j}\mu_{ij}(I_i - I_i^0)\right]\right\}I_j \tag{6a}$$

and some small signal gains G_j

$$\frac{dG_j}{d\vartheta} = \Delta w\langle u_j(x)\rangle + w_0\sum_i \beta_{ij}(I_i^0 - I_i) - \eta G_j - \eta\sum_{im} T_{ijm} I_i G_m. \tag{6b}$$

$\vartheta = t/T$ denotes a rescaled time which is normalized by a characteristic time $T \approx 2.19\ \mu$s of the laser system. In this model the pump rate

$$w = w_0 + \Delta w \tag{7}$$

is assumed to consist of a fixed value w_0 and a variable part Δw that can be used for modulation and control of the laser. The number of active modes depends on the constant pump rate w_0 and we use here a value of $w_0 = 1.247$ where three modes with intensities I_{-1}, I_0, and I_1 are excited. I_j^0 denote fixed point intensities for which the boundary conditions

$$w_0\left\langle\frac{u_j(x)}{1+\sum_i I_i^0 u_i(x)}\right\rangle - 1 - \frac{\epsilon}{\alpha}\left(gI_j^0 + 2\sum_{i\neq j}\mu_{ij}I_i^0\right) = 0 \quad (8)$$

hold ($\langle\cdot\rangle$ denotes spatial averaging). Input signals of the feedback loop are the two infrared (not frequency-doubled) intensities $I_x = I_{-1} + I_1$ and $I_y = I_0$ which are polarized perpendicular to each other.

Figure 3 shows fixed point stabilization obtained with notch filter control applied to the laser model. Here ac components of both infrared intensities I_x and I_y are used as input signals U_{in}^1 and U_{in}^2 of two notch filters (3) with different resonance frequencies f_i and the control signal is given by

$$u(t) = k_1[U_{\text{in}}^1 - U_{\text{out}}^1] + k_2[U_{\text{in}}^2 - U_{\text{out}}^2]. \quad (9)$$

Feedback is switched on at $t = 8$ ms and the system quickly converges to a steady state with constant light intensities.

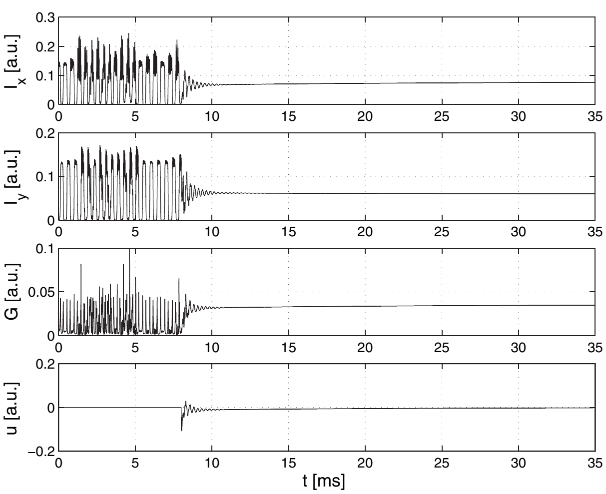

FIG. 3. Simulation of fixed point stabilization of the output power of an intracavity frequency-doubled laser using two notch filters. I_x and I_y denote the infrared intensities whose ac components are used as input signals of the notch filter feedback loop. G is the intensity of the (frequency-doubled) green output light and $u(t)$ the control signal (9). The constant part of the pump rate equals $w_0 = 1.247$ and the control parameters are: $k_1 = 0.9$, $k_2 = 1.7$, $R_1 = 3590\ \Omega$, $R_2 = 420\ \Omega$, $C_1 = C_2 = 25$ mF, and $Q_1 = Q_2 = 0.5$. Control is switched on at $t = 8$ ms and quickly stabilizes the light intensities of the laser with an asymptotically vanishing control signal $u(t)$ (noninvasive control due to high-pass filtering of the input signal).

The dependence of stability regions on the filter quality Q is shown in Fig. 4(a) indicating that too high a filter quality does not provide higher stability. Figure 4(b) shows that stability regions are most enlarged if the feedback gains are chosen different from each other and not too high (reconfirming similar observations made with the Rössler system).

Feedback control using two notch filters has also been applied experimentally using the ac component of the (frequency-doubled) green output intensity as input of the feedback loop. It turned out that the experimental adjustment of the filters is relatively easy if the filter quality is chosen suitably. Figure 5 shows the experimentally measured infrared signals that quickly converge to constant values once control is switched on at $t = 0\ \mu$s. All attempts to stabilize the laser system at this pump level with conventional proportional derivative controllers failed.

In general when trying to stabilize an unstable fixed point of some given dynamical system one has to distinguish between unstable saddles (odd number of positive real eigenvalues) and unstable foci (pairs of unstable complex conjugated eigenvalues). In [9] it was proven that a feedback controller consisting of a single pole high-pass filter is not capable to stabilize unstable steady states with an odd number of real positive eigenvalues (saddle). This holds also for notch filter feedback in its noninvasive form using ac-coupled input signals. To show this we consider the characteristic polynomial

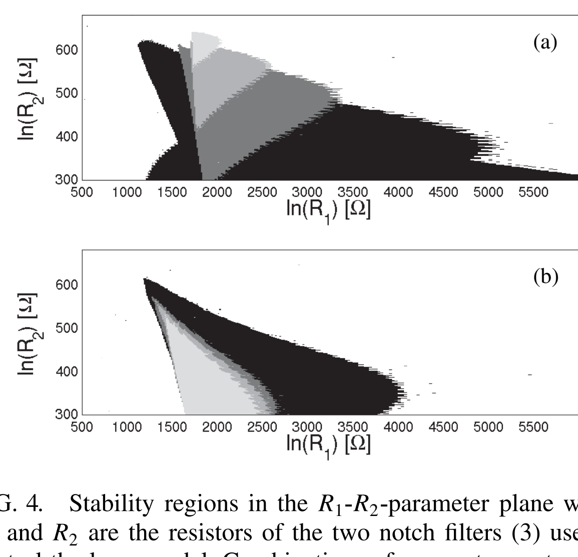

FIG. 4. Stability regions in the R_1-R_2-parameter plane where R_1 and R_2 are the resistors of the two notch filters (3) used to control the laser model. Combinations of parameters not resulting in fixed point stabilization are marked white. (a) Stability region in dependence of filter quality Q for fixed gains $k_1 = 0.9$, $k_2 = 1.7$ from dark to bright: $Q_i = 0.2$; $Q_i = 0.5$; $Q_i = 0.8$; $Q_i = 1.1$; $Q_i = 1.5$. (b) Stability region in dependence of gains k_i for fixed qualities. $Q_1 = 0.5 = Q_2$ from dark to bright: $k_1 = 1.1$, $k_2 = 1.6$; $k_1 = 1.3$, $k_2 = 1.4$; $k_1 = 1.5$, $k_2 = 1.3$; $k_1 = 1.7$, $k_2 = 1.3$; $k_1 = 1.9$, $k_2 = 1.3$.

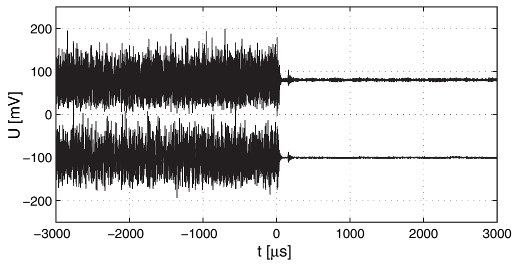

FIG. 5. Experimental fixed point stabilization of the output power of an intracavity frequency-doubled laser using two notch filters. Shown are time series of the infrared intensities and control is switched on at $t = 0\ \mu$s. Input signal for the feedback loop is the ac component of the green intensity emitted by the laser.

$$S(\lambda, k_i) = \det(\lambda I - A) = \lambda^N + \sum_{i=0}^{N-1} a_i \lambda^i \qquad (10)$$

given by the Jacobian matrix A of the full N-dimensional system consisting of the dynamical system and the feedback controller. This polynomial is positive for $\lambda \longrightarrow \infty$ and equals

$$S(0, k_i) = \omega_0^i \prod_{i=1}^{m} \omega_i^2 \prod_{j=1}^{n} (-e_j) \qquad (11)$$

for $\lambda = 0$ if m notch filters are applied with resonance frequencies $\omega_i = 2\pi f_i > 0$ and ac coupling with cutoff frequency $\omega_0 > 0$. Here e_j denote the eigenvalues of the dynamical system to be stabilized. Since an odd number of eigenvalues is positive $S(0, k_i) < 0$ and therefore $S(\lambda, k_i)$ has always at least one eigenvalue in the range $[0, \infty[$ and the system is not stabilizable.

The crucial feature leading to this result is the ac coupling that was introduced to render the control noninvasive. If ac coupling of input signals is not used the proof given above holds no longer and it may be possible to stabilize saddle points with odd numbers of unstable eigenvalues. Whether this is possible or not depends on local properties of the considered dynamical system. A special case are dynamical systems possessing a saddle at the origin. In this case notch filter feedback is noninvasive even without ac-coupled input signals and stabilization is often possible (e.g., for the Lorenz system). This result can be generalized to fixed points away from the origin by using as filter input the difference of the measurement function and the dc level of the fixed point to be stabilized (without employing ac coupling).

To summarize, chaos control using several notch filters in a feedback loop turned out to be a simple but efficient method for stabilizing periodic orbits and steady states. Notch filter feedback is, in particular, useful for fast dynamics because it can easily be implemented in analog hardware and thus provides a promising alternative to delay control methods.

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