Improved Calculation of Electroweak Radiative Corrections and the Value of V_{ud}

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A new method for computing hadronic effects on electroweak radiative corrections to low-energy weak interaction semileptonic processes is described. It employs high order perturbative QCD results originally derived for the Bjorken sum rule along with a large N QCD-motivated interpolating function that matches long- and short-distance loop contributions. Applying this approach to the extraction of the Cabibbo-Kobayashi-Maskawa (CKM) matrix element V_{ud} from superallowed nuclear beta decays reduces the theoretical loop uncertainty by about a factor of 2 and gives $V_{ud} = 0.973\,77(11)(15)(19)$. Implications for CKM unitarity are briefly discussed.

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Precision studies of low-energy semileptonic weak-charged and neutral current processes can be used to test the $SU(3)_c \times SU(2)_L \times U(1)_Y$ standard model at the quantum loop level and to probe for potential "new physics" effects. Examples for which a fraction of a percent experimental sensitivity has already been achieved include pion, neutron, and nuclear beta decays [1], as well as atomic parity violation [2]. In those cases, electroweak radiative corrections (RC) have been computed [3–5] and found to be significant (of order several percent). They must be included in any meaningful confrontation between theory and experiment.

Of course, inherent to any low-energy semileptonic process are uncertainties due to strong interactions, since quarks are involved. To minimize such effects, one often focuses on weak vector current-induced reactions, where CVC (conserved vector current) protects those amplitudes at tree level from strong interaction corrections in the limit of zero momentum transfer. However, even for those amplitudes, electroweak loop corrections can involve weak axial-vector effects not protected by CVC, which give rise to hadronic (strong interaction) uncertainties in their evaluation [3,4]. In this Letter, we focus on the best known and tested examples of that phenomenon, the electroweak radiative corrections to neutron and correspondingly superallowed nuclear beta decays along with their implications for the extraction of the Cabibbo-Kobayashi-Maskawa (CKM) matrix element V_{ud}. However, the method we describe is quite general and can be easily applied to other charged and neutral current semileptonic low-energy reactions of interest in particle, nuclear, and atomic physics.

The extraction of V_{ud} (in fact all CKM matrix elements) entails normalizing a semileptonic reaction rate with respect to the muon lifetime, or equivalently the Fermi constant derived from it,

$$G_\mu = 1.166\,37(1) \times 10^{-5}\ \text{GeV}^{-2}. \qquad (1)$$

For high precision, electroweak radiative corrections to both processes must be included and hadronic as well as environmental effects (e.g., nuclear structure) must be controlled. Toward that end, superallowed ($0^+ \to 0^+$) nuclear beta decay transitions are very special since they involve only the weak vector current at tree level. Small violations of CVC due to the up-down mass difference or nonzero momentum transfer are small $\sim\mathcal{O}(10^{-5})$ and can generally be neglected (or incorporated). Such an analysis leads to the very accurate relationship [6,7]:

$$|V_{ud}|^2 = \frac{2984.48(5)\ \text{s}}{ft(1+\text{RC})} (\text{superallowed } \beta \text{ decays}), \qquad (2)$$

where ft is the product of a phase space statistical decay rate factor f (which depends on the Q value of a specific nuclear beta decay) and its measured half-life t. RC designates the total effect of all radiative corrections relative to muon decay as well as QED-induced nuclear structure isospin violating effects. It is nucleus dependent, ranging from about $+3.1\%$ to $+3.6\%$ for the nine best-measured superallowed decays. So, measuring Q and t combined with computing RC determines V_{ud}. A similar formula will be given later for neutron beta decay. In that case, the Q value $= m_n - m_p$ is very precisely known, but in addition to the neutron lifetime, $g_A \equiv G_A/G_V$ must be accurately measured because both weak axial and vector currents contribute at tree level [1,6].

Our main goal in this Letter is to reduce the hadronic uncertainty in the radiative corrections to superallowed nuclear beta decays and thereby improve the determination of V_{ud}. The need for such an improvement is well illustrated by a survey of ft values and RC for superallowed beta decays by Hardy and Towner [7], more recently updated by Savard *et al.* [8], which found

$$V_{ud} = 0.9736(2)(4)_{EW}, \qquad (3)$$

where the first uncertainty stems primarily from nuclear structure corrections [including $\mathcal{O}(Z^2\alpha^3)$ effects] and very small ft value errors, while the second, dominant error is due to hadronic uncertainties in electroweak loop effects. Although, as we mention later, the first error may currently be an underestimate and the central value of V_{ud} could shift due to future Q value updates, it is clear that the hadronic loop uncertainty, which comes from weak axial-current loop effects, currently limits the determination of V_{ud} and must be improved if further progress is to be made.

Here, we describe a new method for controlling hadronic uncertainties in the radiative corrections to neutron and superallowed nuclear beta decays. It validates our previous results [4,6] increasing V_{ud} by only a small $+0.000\,07$, but reduces the loop uncertainty by about a factor of 2, $(0.0004)_{EW} \longrightarrow (0.0002)_{EW}$ as we now demonstrate.

The one-loop electroweak radiative corrections to the neutron (vector current contribution) and superallowed nuclear beta decays are given by [3,4,9]

$$\mathrm{RC}_{EW} = \frac{\alpha}{2\pi}\left\{\overline{g}(E_m) + 3\ln\frac{m_Z}{m_p} + \ln\frac{m_Z}{m_A} + A_g + 2C_{\mathrm{Born}}\right\}. \tag{4}$$

The first two terms result from loop corrections and bremsstrahlung involving electromagnetic and weak vector current interactions, with $\bar{g}(E_m)$ a universal function [9] integrated over phase space and $3\ln\frac{m_Z}{m_p}$ a short-distance loop effect. They are not affected by strong interactions up to $\mathcal{O}(\frac{\alpha}{\pi}\frac{E_m}{m_p}) \simeq 10^{-5}$ corrections, which can be neglected at our present level of accuracy. Higher order leading logs of order $\alpha^n \ln^n(m_Z/m_p)$, etc., can be summed via a renormalization group analysis [4], and $\mathcal{O}(Z\alpha^2)$ as well as $\mathcal{O}(Z^2\alpha^3)$ contributions have been computed for high Z nuclei [10]. They will not be explicitly discussed here but are included in our final results.

The last three terms in Eq. (4) are induced by weak axial-vector current loop effects. Their primary source is the γW box diagram which involves the current correlator

$$\int d^4x e^{ik\cdot x}\langle p'|T[J_\gamma^\lambda(x)A_W^\rho(0)]|p\rangle, \tag{5}$$

where J_γ^λ and A_W^ρ are the electromagnetic and weak axial-vector currents. That product contains a leading vector current component which contributes to $0^+ \rightarrow 0^+$ nuclear transition elements. Employing the current algebra formulation, one finds [3]

$$Box(\gamma W)_{VA} = \frac{\alpha}{8\pi}\int_0^\infty dQ^2 \frac{m_W^2}{Q^2 + m_W^2} F(Q^2), \tag{6}$$

where Q is a Euclidean loop momentum integration variable.

Previous estimates of Eq. (6) employed the operator product expansion plus lowest order QCD correction to obtain the leading effect [3,4]

$$F(Q^2) \underset{Q^2\to\infty}{\longrightarrow} \frac{1}{Q^2}\left[1 - \frac{\alpha_s(Q^2)}{\pi}\right] + \mathcal{O}\left(\frac{1}{Q^4}\right). \tag{7}$$

Integrating over the range $m_A^2 \leq Q^2 < \infty$ and combining with smaller vertex corrections and box diagrams involving virtual Z and W bosons, that prescription gave a short-distance amplitude contribution

$$\frac{\alpha}{4\pi}\left[\ln\frac{m_Z}{m_A} + A_g\right], \qquad A_g \simeq -0.34. \tag{8}$$

In the numerical estimate, the low-energy cutoff was chosen to be $m_A = 1.2$ GeV, roughly the mass of the A_1 resonance, and the error was estimated by allowing m_A to vary up or down by a factor of 2. Such a heuristic, albeit crude, procedure led to a ± 0.0004 uncertainty in V_{ud}. For the long-distance γW box diagram contribution, nucleon electromagnetic and axial-vector dipole form factors were used to find for neutron decay [4,5]

$$C_{\mathrm{Born}}(\text{neutron}) \simeq 0.8 g_A(\mu_n + \mu_p) \simeq 0.89, \tag{9}$$

where $g_A \simeq 1.27$ and $\mu_n + \mu_p = 0.88$ is the nucleon isoscalar magnetic moment. In the case of superallowed nuclear decays, nuclear quenching modifies C_{Born} (neutron) and nucleon-nucleon electromagnetic effects must be included [11]. Overall, in the case of a neutron, axial-vector-induced one-loop RC to the decay rate amounts to 0.67(8)%. Roughly the same uncertainty $\pm 0.08\%$ applies to superallowed nuclear decays.

To reduce the hadronic uncertainty in RC, we have carried out a new analysis of the γW box diagram axial-vector-induced radiative corrections that incorporates the following $F(Q^2)$ improvements [12]: (1) Short distances: $(1.5\ \mathrm{GeV})^2 \leq Q^2 < \infty$, a domain where QCD corrections remain perturbative.

$$F(Q^2) = \frac{1}{Q^2}\left[1 - \frac{\alpha_s(Q^2)_{\overline{MS}}}{\pi} - C_2\left(\frac{\alpha_s(Q^2)_{\overline{MS}}}{\pi}\right)^2 - C_3\left(\frac{\alpha_s(Q^2)_{\overline{MS}}}{\pi}\right)^3\right], \tag{10}$$

$$C_2 = 4.583 - 0.333 N_F, \tag{11}$$

$$C_3 = 41.440 - 7.607 N_F + 0.177 N_F^2, \tag{12}$$

where N_F equals the number of effective quark flavors. (2) Intermediate distances: $(0.823\ \mathrm{GeV})^2 \leq Q^2 < (1.5\ \mathrm{GeV})^2$.

$$F(Q^2) = \frac{-1.490}{Q^2 + m_\rho^2} + \frac{6.855}{Q^2 + m_A^2} - \frac{4.414}{Q^2 + m_{\rho'}^2}, \tag{13}$$

$$m_\rho = 0.776\ \mathrm{GeV}, \tag{14}$$

$$m_A = 1.230\ \mathrm{GeV}, \tag{15}$$

$$m_{\rho'} = 1.465\ \mathrm{GeV}. \tag{16}$$

(3) Long distances: $0 \leq Q^2 \leq (0.823\ \mathrm{GeV})^2$.

Integrating the long-distance amplitude up to $Q^2 = (0.823\ \mathrm{GeV})^2$, where the integrand matches the interpolating function, and using an update of the nucleon electromagnetic and axial-current dipole form factors, we find

$$C_{\mathrm{Born}}\ (\text{neutron}) \simeq 0.829, \tag{17}$$

a reduction from our own previous result in Eq. (9), where the integration was carried up to $Q^2 = \infty$.

Details of the above calculations will be given in a subsequent publication [12]. Here, we briefly discuss the basis of our improvements along with the results of the above analysis and its implications.

The QCD corrections to the asymptotic form of $F(Q^2)$ have been given in Eq. (10) to $\mathcal{O}(\alpha_s^3)$. The additional terms are identical (in the chiral limit) to QCD corrections to the Bjorken sum rule [13] for polarized electroproduction and can be read off from well-studied calculations [14,15] for that process. Their validity has been well tested experimentally [16]. The equivalence of the QCD corrections to all orders (in the chiral limit) can be easily understood. A chiral transformation $d \to \gamma_5 d$ followed by an isospin rotation in the current correlator of Eq. (5) converts it into the vector-vector correlator responsible for the Bjorken sum rule. Since QCD respects both symmetries in the chiral limit, the QCD corrections must be identical for both cases.

The interpolating function in Eq. (13) is motivated by large N QCD, which predicts it should correspond to an infinite sum of vector and axial-vector resonances [17]. We impose three conditions that determine the residues: (i) The integral of Eqs. (6) and (13) should equal that of Eqs. (6) and (10) in the asymptotic domain $(1.5\ \text{GeV})^2 \le Q^2 \le \infty$, which amounts to a matching requirement between domains 1 and 2. (ii) In the large Q^2 limit, the coefficient of the $1/Q^4$ term in the expansion of Eq. (13) should vanish as required by chiral symmetry [18]. (iii) The interpolator should vanish at $Q^2 = 0$ as required by chiral perturbation theory. Three conditions limit us to three resonances.

The $Q^2 = (0.823\ \text{GeV})^2$ match between domains 2 and 3 was chosen to be the value at which Eq. (13) equals the integrand of the long-distance contribution. Interestingly, that matching occurs near the ρ mass. A novel technical point in the formulation is that in the evaluation of the Feynman diagrams associated with the long-distance contributions the integral over the auxiliary variables is carried out first. This leads to integrands that depend on Q^2 and can therefore be matched with Eq. (13).

Using this approach, we find that at the one-loop electroweak level the last three terms in Eq. (4) are effectively replaced by $2.82\frac{\alpha}{\pi}$ in the case of neutron decay. Comparison with Eqs. (4) and (9) in conjunction with $m_A = 1.2$ GeV, $A_g = -0.34$ shows that in the new formulation these corrections are reduced by 1.4×10^{-4}, which increases V_{ud} by 7×10^{-5}. The smallness of that shift is a validation of our previous result [4,6].

More important than the small reduction in the radiative corrections, our new method provides a more systematic estimate of the hadronic uncertainties as well as experimental verification of its validity [16]. Allowing for a $\pm 10\%$ uncertainty for the C_{Born} correction in Eq. (17), a $\pm 100\%$ uncertainty for the interpolator contribution in the $(0.823\ \text{GeV})^2 \le Q^2 < (1.5\ \text{GeV})^2$ region, and ± 0.0001 uncertainty from neglected higher order effects, we find the total uncertainty in the electroweak radiative corrections is $\simeq \pm 0.00038$, which leads to a $\simeq \pm\ 0.000\,19$ uncertainty in V_{ud}. That corresponds to more than a factor of 2 reduction in the loop uncertainty from hadronic effects.

Employing our new analysis, we find the improved relationship between V_{ud}, the neutron lifetime, and $g_A \equiv G_A/G_V$,

$$|V_{ud}|^2 = \frac{4908.7(1.9)\ \text{s}}{\tau_n(1 + 3g_A^2)}\ \text{(neutron)}. \qquad (18)$$

Future precision measurements of τ_n and g_A used in conjunction with Eq. (18) will ultimately be the best way to determine V_{ud}, but for now it is not competitive [6].

In the case of superallowed ($0^+ \to 0^+$ transitions) nuclear β decays, there are a number of corrections, some nucleus dependent, that must be applied to the ft values. They are collectively called RC in Eq. (2). To make contact with previous studies [1,7], we factorize them as follows:

$$1 + \text{RC} = (1 + \delta_R)(1 - \delta_C)(1 + \Delta). \qquad (19)$$

The first two factors are nucleus dependent, while Δ is roughly nucleus independent, coming primarily from short-distance loop effects. The axial-vector contributions discussed above are included in the product $(1 + \delta_R) \times (1 + \Delta)$, where δ_R includes long-distance radiative corrections as well as nuclear structure effects. Because we include leading logs from higher orders as well as some next-to-leading logs [4,6], the factorization is not exact and Δ will exhibit some small nucleus dependence. The uncertainty in $1 + \delta_R$ comes from $Z^2\alpha^3$ and nuclear structure contributions while a common $\pm 0.03\%$ error in the Coulomb distortion effect is assigned to $1 - \delta_C$.

Employing the corrections given by Hardy and Towner [7,11] along with the results in [6] and our new analysis together with Eqs. (2) and (19) given above leads to the RC and V_{ud} values illustrated in Table I. One finds for the weighted average

$$V_{ud} = 0.973\,77(11)(15)(19)\text{(superallowed } \beta \text{ decays)}. \qquad (20)$$

Comparing with Eq. (3) we see that our analysis gives a somewhat larger V_{ud} due to a $\pm 0.000\,07$ increase from our new prescription along with refinements from Ref. [6] which were not included in Savard *et al.* [8]. Also, Savard *et al.* rounded down in their analysis.

We note that ^{46}V gives a somewhat low value for V_{ud}. It differs from the average by 2.7σ. That particular nucleus recently underwent a Q value revision [8] which lowered its V_{ud}. It may be indicating problems with other Q values. If the other nuclear Q values follow the lead of ^{46}V, we could see a fairly significant reduction in the weighted average for V_{ud}. Clearly, remeasurements of Q values and half-lives of the superallowed decays are highly warranted.

TABLE I. Values of V_{ud} implied by various precisely measured superallowed nuclear beta decays. The ft values are taken from Savard *et al.* [8]. Uncertainties in V_{ud} correspond to (1) nuclear structure and $Z^2\alpha^3$ uncertainties added in quadrature with the ft error [10,11], (2) a common error assigned to nuclear coulomb distortion effects [11], and (3) a common uncertainty from quantum loop effects. Only the first error is used to obtain the weighted average.

Nucleus	ft (s)	1 + RC	V_{ud}
^{10}C	3039.5(47)	1.035 42(36)(30)(38)	0.973 81(77)(15)(19)
^{14}O	3043.3(19)	1.034 41(52)(30)(38)	0.973 68(39)(15)(19)
^{26}Al	3036.8(11)	1.035 82(30)(30)(38)	0.974 06(23)(15)(19)
^{34}Cl	3050.0(12)	1.031 21(38)(30)(38)	0.974 12(26)(15)(19)
^{38}K	3051.1(10)	1.030 99(44)(30)(38)	0.974 04(26)(15)(19)
^{42}Sc	3046.8(12)	1.034 03(54)(30)(38)	0.973 30(32)(15)(19)
^{46}V	3050.7(12)	1.033 76(59)(30)(38)	0.972 80(34)(15)(19)
^{50}Mn	3045.8(16)	1.033 57(67)(30)(38)	0.973 67(41)(15)(19)
^{54}Co	3048.4(11)	1.032 57(75)(30)(38)	0.973 73(40)(15)(19)
		Weighted average	0.973 77(11)(15)(19)

Employing the value of V_{ud} in Eq. (20), the K_{l3} average [19] for V_{us}

$$V_{us} = 0.2257(9)[0.961/f_+(0)], \qquad K_{l3} \text{ average}, \quad (21)$$

with $f_+(0) = 0.961(8)$ [20] and $|V_{ub}|^2 \simeq 1 \times 10^{-5}$ leads to the unitarity test

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9992(5)_{V_{ud}}(4)_{V_{us}}(8)_{f_+(0)}. \quad (22)$$

Good agreement with unitarity (which requires the sum to be 1) is found, with the dominant uncertainty coming now from the theory error in the K_{l3} form factor $f_+(0)$. Equation (22) provides an important test of the standard model at the quantum loop level and a constraint on new physics beyond the standard model at the $\pm 0.1\%$ level. We note, however, that some other calculations [19,21] of $f_+(0)$ and studies of other strangeness changing decays [22] suggest a lower V_{us} value. Combined with further Q value revisions possibly leading to a smaller V_{ud}, they could cause a significant reduction in Eq. (22). A future violation of unitarity is still possible. However, for it to be significant, the theoretical uncertainty in $f_+(0)$ must be further reduced.

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