Staggered Ladder Spectra

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We exactly solve a Fokker-Planck equation by determining its eigenvalues and eigenfunctions: we construct nonlinear second-order differential operators which act as raising and lowering operators, generating ladder spectra for the odd- and even-parity states. The ladders are staggered: the odd-even separation differs from even-odd. The Fokker-Planck equation corresponds, in the limit of weak damping, to a generalized Ornstein-Uhlenbeck process where the random force depends upon position as well as time. The process describes damped stochastic acceleration, and exhibits anomalous diffusion at short times and a stationary non-Maxwellian momentum distribution.

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There are only few physically significant systems with ladder spectra (exactly evenly spaced energy levels). Examples are the harmonic oscillator and the Zeeman-splitting Hamiltonian. In this Letter we introduce and solve a family of eigenvalue problems which occur in an extension of a classic problem in the theory of diffusion, the Ornstein-Uhlenbeck process [1]. Our system is also closely related to a model for stochastic acceleration, introduced by Sturrock [2] in the context of acceleration of charged particles by interstellar fields [3], and analyzed by Golubovic *et al.* [4] (see also Rosenbluth [5]). Our eigenvalue problems have ladder spectra, but they differ from the usual examples in that their spectra consist of two ladders which are staggered; the eigenvalues for eigenfunctions of odd and even symmetry do not interleave with equal spacings. We introduce a new type of raising and lowering operators in our solution, which are nonlinear second-order differential operators. Our generalized Ornstein-Uhlenbeck systems exhibit anomalous diffusion at short times, and non-Maxwellian velocity distributions at equilibrium; we obtain exact expressions which are analogous to results obtained for the standard Ornstein-Uhlenbeck process.

Ornstein-Uhlenbeck processes.—Before we discuss our extension of the Ornstein-Uhlenbeck process, we describe its usual form. This considers a particle of momentum p subjected to a rapidly fluctuating random force $f(t)$ and subject to a drag force $-\gamma p$, so that the equation of motion is $\dot{p} = -\gamma p + f(t)$. The random force has statistics $\langle f(t)\rangle = 0, \langle f(t)f(t')\rangle = C(t-t')$ (angular brackets denote ensemble averages throughout). If the correlation time τ of $f(t)$ is sufficiently short ($\gamma\tau \ll 1$), the equation of motion may be approximated by a Langevin equation: $\mathrm{d}p = -\gamma p dt + dw$, where the Brownian increment dw has statistics $\langle dw\rangle = 0$ and $\langle dw^2\rangle = 2D_0 dt$. The diffusion constant is $D_0 = \frac{1}{2}\int_{-\infty}^{\infty} dt\langle f(t)f(0)\rangle$. This problem is discussed in many textbooks (for example [6]); it is easily shown that the variance of the momentum (with the particle starting at rest) is

$$\langle p^2(t)\rangle = [1-\exp(-2\gamma t)]D_0/\gamma, \tag{1}$$

that the equilibrium momentum distribution is Gaussian, and that the particle (of mass m) diffuses in space with diffusion constant $\mathcal{D}_x = D_0/m^2\gamma^2$.

In many situations the force on the particle will be a function of its position as well as of time. Here we are concerned with what happens in this situation when the damping is weak. We consider a force $f(x,t)$ which has mean value zero, and a correlation function $\langle f(x,t) \times f(x',t')\rangle = C(x-x', t-t')$. The spatial and temporal correlation scales are ξ and τ, respectively. If the momentum of the particle is large compared to $p_0 = m\xi/\tau$, then the force experienced by the particle decorrelates more rapidly than the force experienced by a stationary particle. Thus, if the damping γ is sufficiently weak that the particle is accelerated to a momentum large compared to p_0, the diffusion constant characterizing fluctuations of momentum will be smaller than D_0. The impulse of the force on a particle which is initially at $x=0$ in the time from $t=0$ to $t=\Delta t$ is

$$\Delta w = \int_0^{\Delta t} dt\, f(pt/m, t) + O(\Delta t^2). \tag{2}$$

If Δt is large compared to τ but small compared to $1/\gamma$, we can estimate $\langle \Delta w^2\rangle = 2D(p)\Delta t$, where

$$D(p) = \frac{1}{2}\int_{-\infty}^{\infty} dt\, C(pt/m, t). \tag{3}$$

In the context of undamped stochastic acceleration, a closely related expression was given in [2], and analyzed in [4,5]. When $p \ll p_0$ one recovers $D(p) = D_0$. When $p \gg p_0$, we can approximate (3) to obtain

$$D(p) = \frac{D_1 p_0}{|p|} + O(p^{-2}), \qquad D_1 = \frac{m}{2p_0}\int_{-\infty}^{\infty} dX C(X,0). \tag{4}$$

If the force is the gradient of a potential, $f(x,t) = \partial V(x,t)/\partial x$, then $D_1 = 0$. In this case, expanding the correlation function (assumed to be sufficiently differentiable) in (3) in its second argument gives $D(p) \sim D_3 p_0^3/|p^3|$, where D_3 may be expressed as an integral over the correlation function of $V(x, t)$. To summarize: the momentum diffusion constant is a decreasing function of momentum, such that $D(p) \sim |p|^{-1}$ for a generic random force, or $D(p) \sim |p|^{-3}$ for a gradient force.

Fokker-Planck equation.—The probability density for the momentum, $P(p, t)$, satisfies a Fokker-Planck equation. Following the approach in [6], we obtain $\partial_t P = \partial_p[\gamma p P + D(p)\partial_p P]$. A related equation (without the damping term) was introduced in [2,4,5] and applied to the stochastic acceleration of particles in plasmas [with subsequent contributions concentrating on refining models for $D(p)$, see, for example, [7,8]]. In the following we obtain exact solutions to the Fokker-Planck equation in the cases where $D(p) = D_1 p_0/|p|$ (which we consider first) and $D(p) = D_3 p_0^3/|p|^3$ (treated in the same way and discussed at the end of the Letter).

Introducing dimensionless variables [$t' = \gamma t$ and $z = p(\gamma/p_0 D_1)^{1/3}$], the Fokker-Planck equation for the case where $D(p) \propto |p|^{-1}$ becomes

$$\partial_{t'} P = \partial_z(zP + |z|^{-1}\partial_z P) \equiv \hat{F}P. \qquad (5)$$

It is convenient to transform the Fokker-Planck operator $\hat{F}$ to a Hermitian form, with Hamiltonian

$$\hat{H} = P_0^{-1/2}\hat{F}P_0^{1/2} = 1/2 - |z|^3/4 + \partial_z|z|^{-1}\partial_z, \qquad (6)$$

where $P_0(z) \propto \exp(-|z|^3/3)$ is the stationary solution satisfying $\hat{F}P_0 = 0$. We solve the diffusion problem by constructing the eigenfunctions of the Hamiltonian operator. In the following we make free use of the Dirac notation [9] of quantum mechanics to write the equations in a compact form and to emphasize their structure.

Summary of principal results.—We start by listing our results [for the case of random forcing, where $D(p) \propto 1/|p|$]. We construct the eigenvalues λ_n and eigenfunctions $\psi_n(z)$ of the operator $\hat{H}$. We identify raising and lowering operators $\hat{A}^+$ and $\hat{A}$ which map one eigenfunction to another with, respectively, two more or two fewer nodes. We use these to show that the spectrum of $\hat{H}$ consists of two superposed equally spaced spectra (ladder spectra) for even and odd parity states:

$$\lambda_n^+ = -3n, \qquad \lambda_n^- = -(3n+2), \qquad n = 0, \ldots, \infty. \qquad (7)$$

The spectrum of the Hamiltonian (6) is displayed on the right-hand side of Fig. 1. It is unusual because the odd-even step is different from the even-odd step, due to the singularity of the Hamiltonian at $z = 0$. Our raising and lowering operators allow us to obtain matrix elements required for calculating expectation values, such as the variance of the momentum for a particle starting at rest at $t = 0$:

$$\langle p^2(t)\rangle = \left(\frac{p_0 D_1}{\gamma}\right)^{2/3} \frac{3^{7/6}\Gamma(2/3)}{2\pi}(1 - e^{-3\gamma t})^{2/3}. \qquad (8)$$

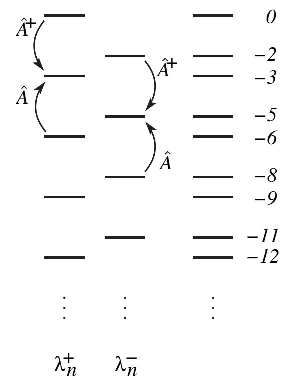

FIG. 1. The spectrum of $\hat{H}$ (right) is the sum of two equally spaced (ladder) spectra λ_n^- and λ_n^+ shifted with respect to each other (left).

This is reminiscent of Eq. (1) for the standard Ornstein-Uhlenbeck process, however (8) exhibits anomalous diffusion for small times. At large times $\langle p^2(t)\rangle$ converges to the expectation of p^2 with the stationary (non-Maxwellian) momentum distribution

$$P_0(p) = \mathcal{N}\exp[-\gamma|p|^3/(3p_0 D_1)] \qquad (9)$$

($\mathcal{N}$ is a normalization constant). At large times the dynamics of the spatial displacement is diffusive $\langle x^2(t)\rangle \sim 2\mathcal{D}_x t$ with diffusion constant

$$\mathcal{D}_x = \frac{(p_0 D_1)^{2/3}}{m^2\gamma^{5/3}} \frac{\pi 3^{-5/6}}{2\Gamma(2/3)^2} F_{32}\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}; \frac{5}{3}, \frac{5}{3}; 1\right) \qquad (10)$$

(here F_{32} is a hypergeometric function). At small times, by contrast, we obtain anomalous diffusion

$$\langle x^2(t)\rangle = C_x[(p_0 D_1)^{2/3} m^{-2}]t^{8/3}, \qquad (11)$$

where the constant C_x is given by (29) below.

Ladder operators and eigenfunctions.—The eigenfunctions of the Fokker-Planck Eq. (5) are alternately even and odd functions, defined on the interval $(-\infty, \infty)$. The operator $\hat{H}$, describing the limiting case of this Fokker-Planck operator, is singular at $z = 0$. We identify two eigenfunctions of $\hat{H}$ by inspection, $\psi_0^+(z) = C_0^+\exp(-|z|^3/6)$, which has eigenvalue $\lambda_0^+ = 0$ and $\psi_0^+(z) = C_0^- z|z| \times \exp(-|z|^3/6)$, with $\lambda_0^- = -2$. These eigenfunctions are of even and odd parity, respectively. Our approach to determining the full spectrum will be to define a raising operator $\hat{A}^+$ which maps any eigenfunction $\psi_n^\pm(z)$ to its successor with the same parity, $\psi_{n+1}^\pm(z)$, having two additional nodes.

We now list definitions of the operators we use: raising and lowering operators, $\hat{A}^\dagger$ and $\hat{A}$, as well as an alternative representation of the Hamiltonian:

$$\hat{a}^\pm = (\partial_z \pm z|z|/2), \qquad \hat{A} = \hat{a}^+|z|^{-1}\hat{a}^+,$$
$$\hat{A}^\dagger = \hat{a}^-|z|^{-1}\hat{a},^- \qquad \hat{H} = \hat{a}^-|z|^{-1}\hat{a}^+, \qquad (12)$$
$$\hat{G} = \hat{a}^+|z|^{-1}\hat{a}^-.$$

Note that $\hat{A}^\dagger$ is the Hermitian conjugate of $\hat{A}$. We have

$$[\hat{H}, \hat{A}] = 3\hat{A} \quad \text{and} \quad [\hat{H}, \hat{A}^\dagger] = -3\hat{A}^\dagger \tag{13}$$

(the square brackets are commutators). These expressions show that the action of $\hat{A}$ and $\hat{A}^\dagger$ on any eigenfunction is to produce another eigenfunction with eigenvalue increased or decreased by three, or else to produce a function which is identically zero. The operator $\hat{A}^\dagger$ adds two nodes, and repeated action of $\hat{A}^\dagger$ on $\psi_0^+(z)$ and $\psi_0^-(z)$ therefore exhausts the set of eigenfunctions. Together with $\lambda_0^+ = 0$ and $\lambda_0^- = -2$ this establishes that the spectrum of $\hat{H}$ is indeed (7). Some other useful properties of the operators of Eq. (12) are

$$[\hat{A}^\dagger, \hat{A}] = 3(\hat{H} + \hat{G}), \qquad \hat{H} - \hat{G} = \hat{I}, \qquad \hat{A}^\dagger\hat{A} = \hat{H}^2 + 2\hat{H}. \tag{14}$$

We represent the eigenfunctions of $\hat{H}$ by kets $|\psi_n^-\rangle$ and $|\psi_n^+\rangle$. The actions of $\hat{A}$ and $\hat{A}^\dagger$ are

$$\hat{A}^\dagger|\psi_n^\pm\rangle = C_{n+1}^\pm|\psi_{n+1}^\pm\rangle, \qquad \hat{A}|\psi_n^\pm\rangle = C_n^\pm|\psi_{n-1}^\pm\rangle, \tag{15}$$

where [using (14)] we have $C_n^\pm = \sqrt{3n(3n \mp 2)}$.

A peculiar feature of $\hat{A}$ and $\hat{A}^\dagger$ is that they are of second order in $\partial/\partial z$, whereas other examples of raising and lowering operators are of first order in the derivative. The difference is associated with the fact that the spectrum is a staggered ladder: only states of the same parity have equal spacing, so that the raising and lowering operators must preserve the odd-even parity. This suggests replacing a first-order operator which increases the quantum number by one with a second-order operator which increases the quantum number by two, preserving parity.

There is an alternative approach to generating the eigenfunctions of $\hat{H}$. This equation falls into one of the classes considered in [10], and we have written down first-order operators which map one eigenfunction into another. However, these operators are themselves functions of the quantum number n, making the algebra cumbersome. We have not succeeded in reproducing our results with the ‘‘Schrödinger factorization’’ method.

Propagator and correlation functions.—The propagator of the Fokker-Planck Eq. (5) can be expressed in terms of the eigenvalues λ_n^σ and eigenfunctions $\phi_n^\sigma(z) = P_0^{-1/2}\psi_n^\sigma(z)$ of $\hat{F}$:

$$K(y, z; t') = \sum_{n=0}^{\infty}\sum_{\sigma=\pm 1} a_n^\sigma(y)\phi_n^\sigma(z)\exp(\lambda_n^\sigma t'). \tag{16}$$

Here y is the initial value and z is the final value of the coordinate. The expansion coefficients $a_n^\sigma(y)$ are determined by the initial condition $K(y, z; 0) = \delta(z - y)$, namely, $a_n^\sigma(y) = P_0^{-1/2}\psi_n^\sigma(y)$. In terms of the eigenfunctions of $\hat{H}$ we have

$$K(y, z; t') = \sum_{n\sigma} P_0^{-1/2}(y)\psi_n^\sigma(y)P_0^{1/2}(z)\psi_n^\sigma(z)\exp(\lambda_n^\sigma t'). \tag{17}$$

The propagator determines correlation functions. Assuming $z_0 = 0$ we obtain for the expectation value of a function $O(z)$ at time t

$$\langle O(z(t'))\rangle = \sum_{n=0}^{\infty}\frac{\psi_n^+(0)}{\psi_0^+(0)}\langle\psi_0^+|O(\hat{z})|\psi_n^+\rangle\exp(\lambda_n^+ t'). \tag{18}$$

Similarly, for the correlation function of $O(z(t_2'))$ and $O(z(t_1'))$ (with $t_2' > t_1' > 0$)

$$\langle O(z(t_2'))O(z(t_1'))\rangle = \sum_{nm\sigma}\frac{\psi_m^+(0)}{\psi_0^+(0)}\langle\psi_0^+|O(\hat{z})|\psi_n^\sigma\rangle \times \langle\psi_n^\sigma|O(\hat{z})|\psi_m^+\rangle \times \exp[\lambda_n^\sigma(t_2' - t_1') + \lambda_m^+ t_1']. \tag{19}$$

Momentum diffusion.—To determine the time-dependence of $\langle p^2(t)\rangle$ we need to evaluate the matrix elements $Y_{0n} = \langle\psi_0^+|\hat{z}^2|\psi_n^+\rangle$. A recursion for these elements is obtained as follows. Let $Y_{0n+1} = \langle\psi_0^+|\hat{z}^2\hat{A}^\dagger|\psi_n^+\rangle/C_{n+1}^+$. Write $\hat{z}\hat{A}^\dagger = \hat{z}\,\hat{G} + \hat{z}(\hat{A}^\dagger - \hat{G}) = \hat{z}(\hat{H} - \hat{I}) + \hat{z}(\hat{A}^\dagger - \hat{G})$. It follows

$$\langle\psi_0^+|\hat{z}^2\hat{A}^\dagger|\psi_n^+\rangle = (\lambda_n^+ - 1)Y_{0n} + \langle\psi_0^+|\hat{z}^2(\hat{A}^\dagger - \hat{G})|\psi_n^+\rangle. \tag{20}$$

Using $(\hat{A}^\dagger - \hat{G}) = -\hat{z}\hat{a}^-$ and $[\hat{z}^3, \hat{a}^-] = -3\hat{z}^2$ we obtain $Y_{0n+1} = \lambda_n^+ + 2Y_{0n}/C_{n+1}^+$, and together with $Y_{00} = 3^{7/6}\Gamma(2/3)/(2\pi)$ this gives

$$Y_{0n} = (-1)^{n+1}\frac{3^{17/12}\Gamma(2/3)^{3/2}}{\sqrt{2}\pi^{3/2}(3n - 2)}\frac{\sqrt{\Gamma(n + 1/3)}}{\sqrt{\Gamma(n + 1)}}. \tag{21}$$

We also find

$$\psi_n^+(0)/\psi_0^+(0) = (-1)^n\sqrt{\frac{\sqrt{3}\Gamma(2/3)}{2\pi}\frac{\Gamma(n + 1/3)}{\Gamma(n + 1)}}, \tag{22}$$

and after performing the sum in (18) we return to dimensional variables. The final result is (8).

Spatial diffusion.—The time-dependence of $\langle x^2(t)\rangle$ is determined in a similar fashion, from

$$\langle x^2(t)\rangle = \frac{1}{\gamma^2}\left(\frac{p_0 D_1}{\gamma}\right)^{2/3}\frac{1}{m^2}\int_0^{t'} dt_1'\int_0^{t'} dt_2'\langle z_{t_1'} z_{t_2'}\rangle. \tag{23}$$

The matrix elements $Z_{mn} = \langle\psi_m^+|\hat{z}|\psi_n^-\rangle$ are found by a recursion method, analogous to that yielding (21)

$$Z_{mn} = (-1)^{m-n}\frac{3^{5/6}}{6\pi}(m + n + 1)\Gamma(2/3) \times \frac{\sqrt{\Gamma(n + 1)\Gamma(m + 1/3)}}{\sqrt{\Gamma(m + 1)\Gamma(n + 5/3)}}\frac{\Gamma(n - m + 1/3)}{\Gamma(n - m + 2)} \tag{24}$$

for $l \geq m - 1$ and zero otherwise. Using (19) we obtain

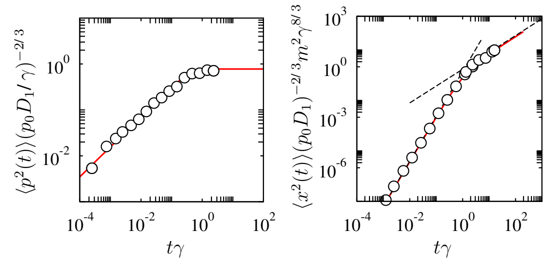

FIG. 2 (color online). Shows $\langle p^2(t)\rangle$ and $\langle x^2(t)\rangle$. Computer simulation of the equations of motion $\dot{p} = -\gamma p + f(x, t)$ and $m\dot{x} = p$ (symbols); theory, Eqs. (9) and (25), red lines. Also shown are the limiting behaviors for $\langle x^2(t)\rangle$, (10) and (11), at long and short times (dashed lines). In the simulations, $C(X, t) = \sigma^2 \exp[-X^2/(2\xi^2) - t^2/(2\tau^2)]$. The parameters were $m = 1$, $\gamma = 10^{-3}$, $\xi = 0.1$, $\tau = 0.1$, and $\sigma = 20$.

$$\langle x^2(t)\rangle = \frac{(p_0D_1)^{2/3}}{m^2\gamma^{5/3}} \sum_{k=0}^{\infty} \sum_{l=k-1}^{\infty} A_{kl}T_{kl}(t') \tag{25}$$

with $A_{kl} = [\psi_k^+(0)/\psi_0^+(0)]Z_{0l}Z_{kl}$ and with

$$T_{kl}(t') = \int_0^{t'} dt_1' \int_{t_1'}^{t'} dt_2' e^{\lambda_l^-(t_2'-t_1')+\lambda_k^+ t_1'} + \int_0^{t'} dt_1' \int_0^{t_1'} dt_2' e^{\lambda_l^-(t_1'-t_2')+\lambda_k^+ t_2'}. \tag{26}$$

We remark upon an exact sum rule for the A_{kl}, and also on their asymptotic form for $k \gg 1$, $l \gg 1$:

$$\sum_{k=0}^{l+1} A_{kl} = 0, \qquad A_{kl} \sim \frac{\Gamma(2/3)^2}{3^{1/3}4\pi^2} \frac{k+l}{k^{2/3}l^{4/3}(l-k)^{5/3}}. \tag{27}$$

We now show how to derive the limiting behaviors (10) and (11), shown as dashed lines in Fig. 2. At large time x evolves diffusively: $\langle x^2\rangle \sim 2\mathcal{D}_x t$, with the diffusion constant obtained from

$$\mathcal{D}_x = \frac{1}{2m^2\gamma}\left(\frac{p_0D_1}{\gamma}\right)^{2/3} \lim_{T\to\infty} \int_{-\infty}^{\infty} dt' \langle z_T z_{t'+T}\rangle = \frac{-1}{m^2\gamma}\left(\frac{p_0D_1}{\gamma}\right)^{2/3} \sum_{n=0}^{\infty} \frac{Z_{0n}^2}{\lambda_n^-}, \tag{28}$$

which evaluates to (10). At small values of t' the double sum (25) is dominated by the large-k, l terms. We evaluate the small-t' behavior by approximating the sums by integrals, using the asymptotic form for the coefficients A_{kl}. A nonintegrable divergence of $A(k, l)$ (as $k \to l$) can be canceled by using the sum rule in Eq. (27). We obtain the limiting behavior (11) with

$$C_x = -C \int_0^{\infty} \frac{dx}{x^{8/3}} \int_0^1 dy \left[\frac{a(x) - a(xy)}{1-y} - xa'(x)\right] b(y), \tag{29}$$

where $a(x) = [1 - \exp(-x)]/x$, $b(y) = (1+y) \times (1-y)^{-5/3} y^{-2/3}$, and $C = 3^{1/3}\Gamma(2/3)^2/(2\pi^2)$. The integral is convergent and can be evaluated numerically to give $C_x = 0.57\ldots$. This is in good agreement with a numerical evaluation of the sum (25), as shown in Fig. 2.

Gradient-force case.—When the force is the gradient of a potential function, we have (generically) $D(p) = D_3 p_0^3/|p|^3 + O(p^{-4})$ [4]. In dimensionless variables the Fokker-Planck equation is $\partial_{t'}P = \partial_z(zP + |z|^{-3}\partial_z P) \equiv \hat{F}P$ instead of (5). This Fokker-Planck equation has the non-Maxwellian equilibrium distribution $P_0(z) = \exp(-|z|^5/5)$. The raising and lowering operators are of the form $\hat{A}^\dagger = \hat{a}^-|z|^{-3}\hat{a}^-$ and $\hat{A} = \hat{a}^+|z|^{-3}\hat{a}^+$ with $\hat{a}^\pm = (\partial_z \pm z|z|^3/2)$. The analogue of (13) is $[\hat{H}, \hat{A}] = 5\hat{A}$, $[\hat{H}, \hat{A}^\dagger] = -5\hat{A}^\dagger$, and the eigenvalues are $0, -4, -5, -9, -10, -14, -15, \ldots$. In this case, too, a closed expression, for example, for $\langle p^2(t)\rangle$ is obtained, analogous to (1) but exhibiting anomalous diffusion

$$\langle p^2(t)\rangle = p_0^2\left(\frac{5D_3}{\gamma p_0^2}\right)^{2/5} \frac{\sin(\pi/5)\Gamma(3/5)\Gamma(4/5)}{\pi} \times (1 - e^{-5\gamma t})^{2/5}. \tag{30}$$

The short-time anomalous diffusion is consistent with the scaling obtained in [4,5] for undamped stochastic acceleration.

Nondifferentiable correlation functions.—The case where $D(p) \propto |p|^{-\zeta}$ (for some general exponent $\zeta > 0$) can be relevant when the correlation function $C(x, t)$ is nonanalytic at $t = 0$. Here too we find raising and lowering operators and staggered ladder spectra and obtain results analogous to those quoted above. The anomalous-diffusion exponents at short times are $\langle p^2(t)\rangle \sim t^{2/(2+\zeta)}$ and $\langle x^2(t)\rangle \sim t^{(6+2\zeta)/(2+\zeta)}$.

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