Aspects of the Confinement Mechanism in Coulomb-Gauge QCD

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Phenomenological consequences of the infrared singular, instantaneous part of the gluon propagator in the Coulomb gauge are investigated. The corresponding quark Dyson-Schwinger equation is solved, neglecting retardation and transverse gluons and regulating the resulting infrared singularities. While the quark propagator vanishes as the infrared regulator goes to zero, the frequency integral over the quark propagator stays finite and well defined. Solutions of the homogeneous Bethe-Salpeter equation for the pseudoscalar and vector mesons as well as for scalar and axial-vector diquarks are obtained. In the limit of a vanishing infrared regulator the diquark masses diverge, while meson properties and diquark radii remain finite and well defined. These features are interpreted with respect to the resulting aspects of confinement for colored quark-quark correlations.

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The substructure of the nucleon has been determined to an enormous precision leaving no doubt that the parton picture emerges from quarks and gluons, the elementary fields of quantum chromodynamics (QCD). Although they are the "elementary particles" of strong interactions, quarks and gluons have never been detected outside hadrons. This phenomenon is called confinement. Despite its importance for particle physics and for an axiomatic approach to quantum field theory our understanding of confinement is far from being satisfactory.

In this Letter we concentrate on certain aspects of confinement for colored composite states. We start from the commonly accepted Wilson criterion [1] and an inequality between the gauge-invariant quark-antiquark potential $V_W(R)$ and the color-Coulomb potential $V_C(\vec{x})$ [2]. The latter quantity is the instantaneous part of the time-time component of the gluon propagator in Coulomb gauge: $D_{00}(\vec{x}, t) \propto V_C(\vec{x})\delta(t)$ + noninst. terms. In Ref. [2] it was shown that if $V_W(R)$ is confining, i.e., if $\lim_{R\to\infty} V_W(R) \to \infty$, then also $|V_C(\vec{x})|$ is confining. This was confirmed in an $SU(2)$ lattice calculation [3] where it was found that $-V_C(\vec{x})$ rises linearly with $R = |\vec{x}|$. However, the corresponding string tension, σ_c, was extracted to be several times the asymptotic one. (If the same holds for the physical case of three colors one infers $\sqrt{\sigma_c} \approx 600\ldots750$ MeV from the generally used value $\sqrt{\sigma_c} \approx 440$ MeV. Note, however, that this increase is not sufficient to resolve the problem of a too small value of the pion decay constant [4], when only a confining potential is used and noninstantaneous interactions, in particular, transverse gluons, are neglected.)

A well-suited formalism for the study of composite or bound states of quarks is the Dyson-Schwinger–Bethe-Salpeter approach [5]. While corresponding investigations in Coulomb gauge, e.g., [6,7], predate those based on model studies employing Landau-gauge QCD Green functions, the latter have been much more numerous and the corresponding studies explore a large number of hadron observables, see, e.g., Refs. [8–13] and references therein. Note that in Landau-gauge QCD the structure of the quark-gluon vertex [14] is an issue of current debate due to its importance for the quark propagator [15].

In this Letter we report on a study of mesons and two-quark composite states employing the color-Coulomb potential $V_C(\vec{x})$ and thus some of the basic features of Coulomb-gauge QCD. We build on investigations of the gluon propagator [16] and the dynamical breaking of chiral symmetry [4,7,17] in Green-function approaches and related results of lattice calculations [3,18]. Our focus is the realization of confinement for quarks and two-quark composite states ("diquarks").

First, we briefly review the quark Dyson-Schwinger (gap) and bound-state Bethe-Salpeter equations. All calculations are performed in Minkowski space. The QCD gap equation determines the quark self-energy due to gluons. It is of the form

$$iS^{-1}(p) = \not{p} - m - \Sigma(p), \quad (1)$$

where $S(p)$ is the renormalized dressed quark propagator, m the current-quark mass, and $\Sigma(p)$ is the quark self-energy. A quark-antiquark bound state is described by the Bethe-Salpeter equation (BSE), which in its homogeneous form is written as (for simplicity we neglect Dirac, flavor, and color indices)

$$\Gamma(P, q) = \int d^4k K(q, k, P) S(k_+) \Gamma(P, k) S(k_-), \quad (2)$$

where P and q are the quark-antiquark pair's total and relative four-momenta, $\Gamma(P, q)$ is the bound state's Bethe-Salpeter amplitude (BSA), $k_\pm = k \pm P/2$ are the individual quark- and antiquark-momenta, and $K(q, k, P)$ is the quark-antiquark scattering kernel. Note that the result of Eq. (1) appears as input in Eq. (2).

The quark self-energy in Eq. (1) is a functional of the quark and gluon propagators and the quark-gluon vertex; a self-consistent solution would require us to simultaneously solve the Dyson-Schwinger equations for these functions and the quark-antiquark scattering kernel from Eq. (2) as well. However, these equations again involve higher Green functions and therefore a truncation of this infinite coupled system of integral equations is necessary.

In the present study we use Coulomb gauge together with an instantaneous approximation and neglect the effects of transverse gluons. These approximations simplify the technical challenges involved in concrete calculations. On the other hand, some component of the physics contained in the system is lost. The results are qualitatively, but not quantitatively significant. We therefore refrain from using physical dimensions, but instead present the quantities in all graphs in appropriate units of the Coulomb string tension σ_c. The reason for the qualitative reliability of the calculations is that the underlying symmetries of the theory are incorporated in the model via Slavnov-Taylor or Ward-Takahashi identities. One important example is the axial-vector Ward-Takahashi identity, which is used to ensure that the kernels of the gap and Bethe-Salpeter equations for pseudoscalar states are related in such a way that chiral symmetry and its dynamical breaking are respected by the truncation. Here, corresponding to the rainbow approximation in the quark Dyson-Schwinger equation, we employ the ladder approximation in the qq scattering kernel in the BSE. In particular, this leads to the correct behavior of the pion mass as a function of the current-quark mass in the chiral limit. This behavior is shown in Fig. 1. (Note: The results in all figures except Fig. 1 are presented for the chiral limit, i.e., zero current-quark mass. The results for finite current-quark mass are analogous.) In this way one can reliably make qualitative statements about hadrons and their properties; however, it is still important to investigate the contributions from retardation effects and transverse gluons, and such efforts are currently made.

In our model the quark self-energy $\Sigma(p)$ in Eq. (1) takes the form

$$\Sigma(p) = C_f 6\pi \int \frac{d^4q}{(2\pi)^4} V_C(\vec{k})\gamma_0 S(q)\gamma_0, \tag{3}$$

where $C_f = (N_c^2 - 1)/(2N_c) = 4/3$ and $\vec{k} = \vec{p} - \vec{q}$. Our particular choice for the color-Coulomb-potential $V_C(\vec{k})$ will be given in Eq. (8). In the following we will use p to denote $p = |\vec{p}|$. The q_0 integration in Eq. (3) can be performed easily. One makes the ansatz $S^{-1}(p) := -i[\gamma_0 p_0 - \vec{\gamma}\cdot\vec{p}C(p) - B(p)]$ and obtains two coupled integral equations for the functions $B(p)$ and $C(p)$

$$B(p) = m + \frac{1}{2\pi^2}\int d^3q V_C(k)\frac{M(q)}{\tilde{\omega}(q)} \tag{4}$$

$$C(p) = 1 + \frac{1}{2\pi^2}\int d^3q V_C(k)\hat{p}\cdot\hat{q}\frac{q}{p\tilde{\omega}(q)}, \tag{5}$$

where $\hat{p} = \vec{p}/p$, m is the current-quark mass, $\tilde{\omega}(p) := \sqrt{M^2(p) + p^2}$, and $M(q) := B(q)/C(q)$ is the quark "mass function." Its infrared behavior is a result of dynamical chiral symmetry breaking and can be used to define a constituent-quark mass; we have plotted the mass function M as a function of q^2 in Fig. 2 (details of this figure will be specified below).

The same approximations and conventions are used in the BSE. For pseudoscalar mesons in our model (and correspondingly scalar diquarks) the BSA can be characterized in terms of two scalar functions $h(p)$ and $g(p)$, which essentially are the coefficients of the pseudoscalar and axial-vector structures in the BSA. For details, see Ref. [19]. The BSE, Eq. (2), in terms of $h(p)$ and $g(p)$ in our model becomes

$$h(p)\omega(p) = \frac{1}{2\pi^2}\int d^3q V_C(k)\left[h(q) + \frac{m_\pi^2}{4\omega(q)}g(q)\right], \tag{6}$$

$$g(p)\left[\omega(p) - \frac{m_\pi^2}{4\omega(p)}\right] = h(p) + \frac{1}{2\pi^2}\int d^3q V_C(k) \times\left[\frac{M(p)M(q) + \vec{p}\cdot\vec{q}}{\tilde{\omega}(p)\tilde{\omega}(q)}\right]g(q), \tag{7}$$

where m_π is the bound state's (e.g., the pion's) yet unknown mass and $\omega(p) = C(p)\tilde{\omega}(p)$.

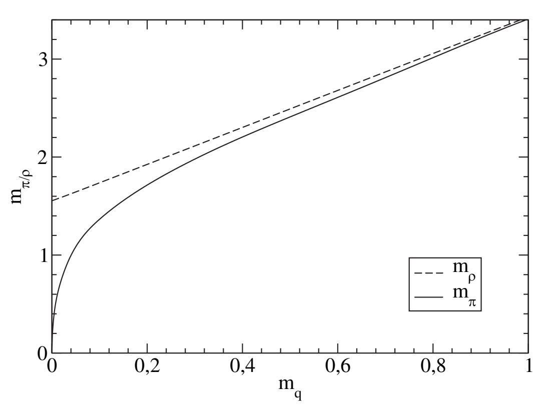

FIG. 1. The pion and rho masses as functions of the current-quark mass in the limit $\mu_{\rm IR} \to 0$. All quantities are given in appropriate units of $\sqrt{\sigma_c}$, σ_c being the Coulomb string tension (see text).

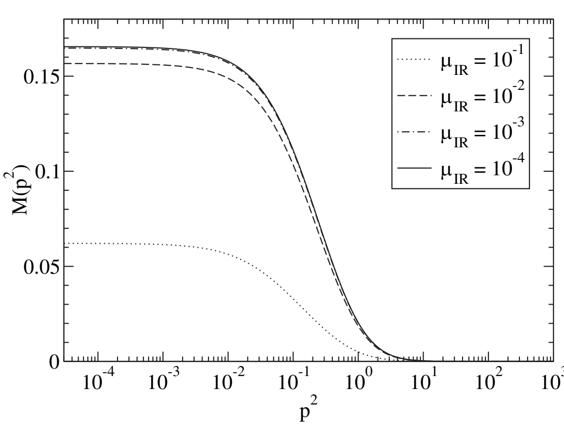

FIG. 2. The quark mass function $M(q^2)$ for four values of the infrared regulator $\mu_{\rm IR}$ in the chiral limit $m = 0$. All quantities are given in appropriate units of $\sqrt{\sigma_c}$.

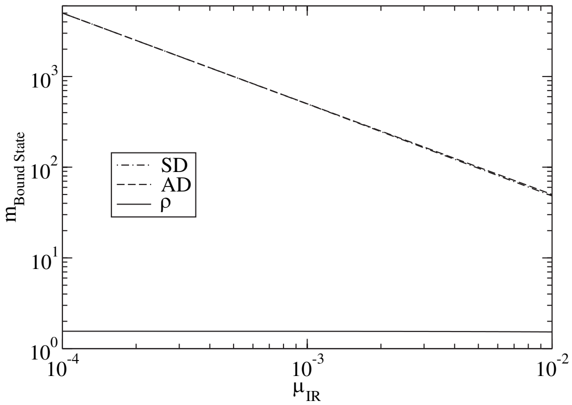

FIG. 3. The masses of the ρ as well as the scalar (SD) and axial-vector (AD) diquarks as functions of the infrared regulator μ_{IR} in the chiral limit. The mass of the π is identically zero for all values of μ_{IR} and therefore not shown in the graph. All quantities are given in appropriate units of $\sqrt{\sigma_c}$.

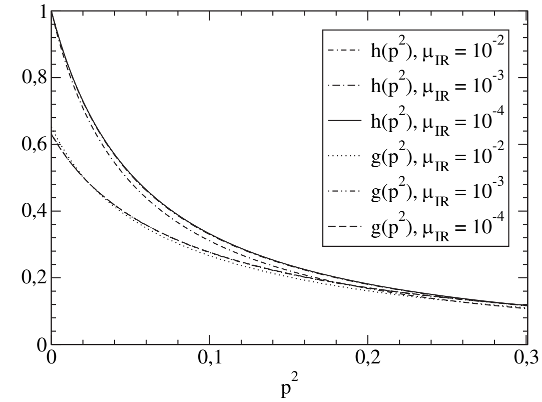

FIG. 4. Pion Bethe-Salpeter amplitude components g and h as functions of the infrared regulator μ_{IR}. For convenience, the amplitudes are normalized such that $h(0) = 1$. All quantities are given in appropriate units of $\sqrt{\sigma_c}$.

For vector mesons (and correspondingly axial-vector diquarks) the BSA has four linearly independent amplitudes. The construction of the four coupled integral equations corresponding to the BSE is analogous to the pseudoscalar case.

The Coulomb-gluon part V_C of the interaction in Eqs. (4)–(6) is chosen to be

$$V_C(k) = \frac{\sigma_c}{(k^2)^2}, \qquad (8)$$

where σ_c is the Coulomb string tension. Obviously, $V_C(k)$ is infrared singular. It is regulated by a parameter μ_{IR} such that the momentum dependence is modified to

$$V_C(k) = \frac{\sigma_c}{(k^2)^2} \rightarrow \frac{\sigma_c}{(k^2 + \mu_{\mathrm{IR}}^2)^2}. \qquad (9)$$

In this fashion all quantities and observables become μ_{IR} dependent and one obtains the final result for some $f(\mu_{\mathrm{IR}})$ by taking the limit $f = \lim_{\mu_{\mathrm{IR}}\to 0} f(\mu_{\mathrm{IR}})$. This is illustrated for the quark mass function in Fig. 2: $M(p^2)$ is plotted for different values of μ_{IR} and it is clear that the curves converge onto a final result for $\mu_{\mathrm{IR}} \to 0$.

In order to check the UV behavior one can use a Richardson potential [20], which has the momentum dependence $V_C(k) \sim 1/[k^2 \ln(1 + k^2/\Lambda^2)]$. The advantage of our choice for V_C is that the angular integration required to solve Eqs. (4) and (5) can be performed analytically. We have checked that the qualitative results presented in this Letter can be reproduced with the Richardson potential. Details of this approach and its UV renormalization will be published elsewhere.

The homogeneous BSE in Eq. (2) is solved by introducing an eigenvalue $\lambda(P^2 = M^2)$ with M the bound-state mass. One then finds M such that $\lambda = 1$ (for mesons) and $\lambda = 2$ (for diquarks). For details, see, e.g., [12].

The curve for $\lambda(P^2)$ gets less inclined with smaller values of the infrared regulator μ_{IR}, and its intersection point with $\lambda = 1$ stabilizes in the limit $\mu_{\mathrm{IR}} \to 0$. As a consequence, while the meson mass is stable, the mass eigenvalue for the corresponding diquark state [corresponding to $\lambda(M) = 2$] increases like $1/\mu_{\mathrm{IR}}$, ultimately completely removing these states from the physical spectrum. We have illustrated these effects in Fig. 3 for values of $10^{-4} \le \mu_{\mathrm{IR}} \le 10^{-2}$.

Note: for $N_c = 2$ diquarks correspond to baryons. In particular, in ladder approximation the respective color factors for meson and baryon BSEs are identical. Therefore, the properties of the scalar (axial-vector) baryon are identical to those of the pion (ϱ meson). For $N_c \ge 3$ the ratio of quark-quark to quark-antiquark color factors increases like $N_c - 1$; this means that the argument given above is also valid in the large-N_c limit.

We studied the BSAs as $\mu_{\mathrm{IR}} \to 0$: the results for g and h (6) are presented in Figs. 4 and 5 for the pion and scalar diquark, respectively. For convenience, the normalization of the amplitudes has been chosen such that $h(0) = 1$. We note, however, that IR cancellations appearing in the pion case lead to a stable h as well as ratio of g/h, which is not the case (as one would naively expect) in the diquark case: there $g/h \sim \mu_{\mathrm{IR}} \to 0$ and $h \sim 1/\sqrt{\mu_{\mathrm{IR}}}$. Still, one can in-

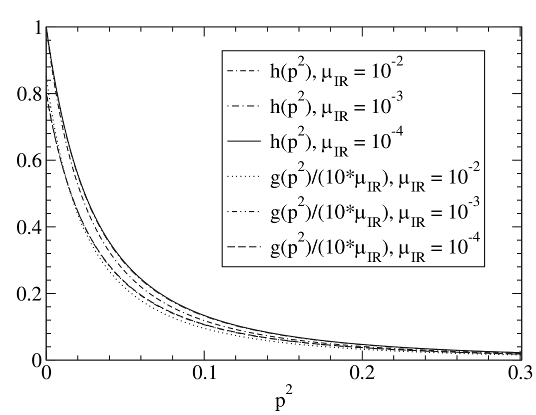

FIG. 5. Same as Fig. 4 for the scalar diquark.

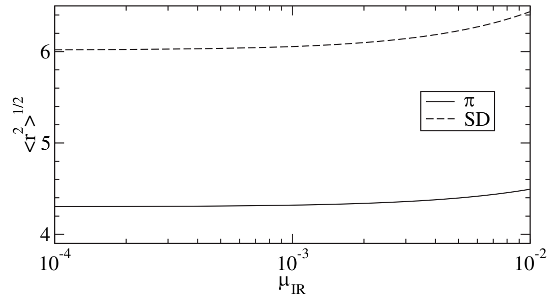

FIG. 6. Charge radii for π meson as well as SD as functions of the infrared regulator $\mu_{\rm IR}$. All quantities are given in appropriate units of $\sqrt{\sigma_c}$.

vestigate the charge radii for both meson and diquark states by requesting that the electromagnetic form factor at the origin yields the bound-state charge, which gives finite results in the limit $\mu_{\rm IR} \to 0$. Plots of the pion and scalar diquark charge radii are shown in Fig. 6. The results for vector-meson and axial-vector-diquark amplitudes are analogous.

We have performed a study of pseudoscalar- and vector-meson states and their corresponding diquark partners in a simple model of Coulomb-gauge QCD in the context of Dyson-Schwinger equations, which allows for obtaining reliable qualitative information about hadrons. The infrared singularities in the integrands are regulated by the scale $\mu_{\rm IR}$ such that final results are obtained in the limit $\mu_{\rm IR} \to 0$. In this limit the masses and charge radii for the mesons are stable; for their diquark partners only the masses diverge like $1/\mu_{\rm IR}$, while the charge radii do not. Thus the diquarks are removed from the physical spectrum reflecting confinement of colored quark-quark correlations. Nevertheless they possess a well-defined size. This adds to the motivation of nucleon studies in a covariant quark-diquark picture [10,11].

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[1] K. G. Wilson, Phys. Rev. D **10**, 2445 (1974).
[2] D. Zwanziger, Phys. Rev. Lett. **90**, 102001 (2003).
[3] J. Greensite, S. Olejnik, and D. Zwanziger, Phys. Rev. D **69**, 074506 (2004).
[4] S.L. Adler and A.C. Davis, Nucl. Phys. **B244**, 469 (1984); R. Alkofer and P.A. Amundsen, Nucl. Phys. **B306**, 305 (1988).
[5] C. D. Roberts and S. M. Schmidt, Prog. Part. Nucl. Phys. **45**, S1 (2000); R. Alkofer and L. von Smekal, Phys. Rep. **353**, 281 (2001); P. Maris and C. D. Roberts, Int. J. Mod. Phys. E **12**, 297 (2003).
[6] A. Le Yaouanc *et al.*, *Hadron Transitions in the Quark Model* (Gordon and Breach, New York 1988) and references therein.
[7] J.R. Finger and J.E. Mandula, Nucl. Phys. **B199**, 168 (1982); J. Govaerts, J.E. Mandula, and J. Weyers, Nucl. Phys. **B237**, 59 (1984).
[8] P. Maris and C. D. Roberts, Phys. Rev. C **56**, 3369 (1997); P. Maris and P. C. Tandy, Phys. Rev. C **60**, 055214 (1999); A. Holl, A. Krassnigg, and C. D. Roberts, Phys. Rev. C **70**, 042203 (2004).
[9] P. Maris and P. C. Tandy, Nucl. Phys. **A663**, 401 (2000); Phys. Rev. C **61**, 045202 (2000); **62**, 055204 (2000); A. Holl, A. Krassnigg, P. Maris, C. D. Roberts, and S. V. Wright, Phys. Rev. C **71**, 065204 (2005).
[10] M. Oettel, G. Hellstern, R. Alkofer, and H. Reinhardt, Phys. Rev. C **58**, 2459 (1998); M. Oettel, R. Alkofer, and L. von Smekal, Eur. Phys. J. A **8**, 553 (2000); M. Oettel, L. von Smekal, and R. Alkofer, Comput. Phys. Commun. **144**, 63 (2002).
[11] J. C. R. Bloch, A. Krassnigg, and C. D. Roberts, Few-Body Syst. **33**, 219 (2003); R. Alkofer, A. Höll, M. Kloker, A. Krassnigg, and C. D. Roberts, Few-Body Syst. **37**, 1 (2005); A. Höll, R. Alkofer, M. Kloker, A. Krassnigg, C. D. Roberts, and S. V. Wright, Nucl. Phys. **A755**, 298 (2005).
[12] P. Maris, Few-Body Syst. **32**, 41 (2002); P. Maris, nucl-th/0412059.
[13] R. Alkofer, P. Watson, and H. Weigel, Phys. Rev. D **65**, 094026 (2002); P. Watson and W. Cassing, Few-Body Syst. **35**, 99 (2004); P. Watson, W. Cassing, and P. C. Tandy, Few-Body Syst. **35**, 129 (2004).
[14] J. Skullerud and A. Kizilersu, J. High Energy Phys. 09 (2002) 013; M. S. Bhagwat, A. Holl, A. Krassnigg, C. D. Roberts, and P. C. Tandy, Phys. Rev. C **70**, 035205 (2004); F. J. Llanes-Estrada, C. S. Fischer, and R. Alkofer, hep-ph/0407332; M. S. Bhagwat and P. C. Tandy, Phys. Rev. D **70**, 094039 (2004).
[15] R. Alkofer, W. Detmold, C. S. Fischer, and P. Maris, Phys. Rev. D **70**, 014014 (2004); C. S. Fischer and R. Alkofer, Phys. Rev. D **67**, 094020 (2003).
[16] A. P. Szczepaniak, Phys. Rev. D **69**, 074031 (2004); A. P. Szczepaniak and E. S. Swanson, AIP Conf. Proc. **549**, 330 (2000); Phys. Rev. D **65**, 025012 (2002); D. Zwanziger, Phys. Rev. D **70**, 094034 (2004); C. Feuchter and H. Reinhardt, Phys. Rev. D **70**, 105021 (2004).
[17] F. J. Llanes-Estrada, S. R. Cotanch, A. P. Szczepaniak, and E. S. Swanson, Phys. Rev. C **70**, 035202 (2004).
[18] A. Cucchieri and D. Zwanziger, Phys. Rev. D **65**, 014001 (2002); J. Greensite, Prog. Part. Nucl. Phys. **51**, 1 (2003); J. Greensite and S. Olejnik, Phys. Rev. D **67**, 094503 (2003); J. Greensite, S. Olejnik, and D. Zwanziger, hep-lat/0410028.
[19] K. Langfeld, R. Alkofer, and P. A. Amundsen, Z. Phys. C **42**, 159 (1989).
[20] J. L. Richardson, Phys. Lett. B **82**, 272 (1979).