Proposed Experiments to Probe the Non-Abelian $\nu = 5/2$ Quantum Hall State

Ady Stern[1] and Bertrand I. Halperin[2]

[1]Department of Condensed Matter Physics, The Weizmann Institute of Science, Rehovot 76100, Israel
[2]Physics Department, Harvard University, Cambridge, Massachusetts 02138, USA

(Received 19 August 2005; published 6 January 2006)

We propose several experiments to test the non-Abelian nature of quasiparticles in the fractional quantum Hall state at $\nu = 5/2$. In a simplified version of the experiment suggested by [S. Das Sarma, M. Freedman, and C. Nayak, Phys. Rev. Lett. **94**, 166802 (2005).], interference is turned on and off when the number of localized quasiparticles between the interfering paths varies between even and odd. We find analogous effects in the thermodynamic properties of closed systems.

DOI: 10.1103/PhysRevLett.96.016802 PACS numbers: 73.43.Fj, 73.43.Jn

Non-Abelian quantum Hall states have been the focus of much theoretical interest since their proposal by Moore and Read [1–3]. This interest has been recently revived for several reasons. First, improved experimental capabilities allow for better inspection of the quantum Hall states in the range of Landau level filling fractions of $2 < \nu < 4$, where at least some of the observed states may be non-Abelian [4]. Second, non-Abelian quantum Hall states are believed to be abundant in rotated Bose-Einstein condenstaes, at high (presently unattainable) angular rotation velocity [5,6]. And third, non-Abelian quantum Hall states are prime candidates for the realization of a topological quantum computer [7].

Experimental study of the non-Abelian nature of quantum Hall states has so far been lacking, both because of difficulty in reaching the required experimental conditions (particularly the quality of the two dimensional electron gas), and because of the lack of proposals for realizable experimental tests. The quality of samples was impressively improved in recent years, and the need for proposals for experiments becomes ever more burning. Important steps in that direction were carried out by Fradkin *et al.* [8], who considered an interferometer for non-Abelian quasiparticles and pointed out its general relation to Jones polynomials, and by Das Sarma *et al.*, who proposed an interference experiment whose results test the non-Abelian nature of excitations in the $\nu = 5/2$ state [9].

In this Letter we propose several simplified versions for such an experiment, and examine the conditions under which it may indeed be a test for the non-Abelian nature of the $\nu = 5/2$ state. We focus first on a Hall bar where two quantum point contacts introduce weak backscattering of current, with amplitudes of t_L and t_R respectively (see Fig. 1). In the simplest case, where the bulk is in an integer quantum Hall state, one expects the backscattered current to be proportional to $|t_L + e^{i2\pi\Omega} t_R|^2$ where the relative phase Ω is the number of flux quanta enclosed in the island defined by the two quantum point contacts and the two edges connecting them. This phase can then be varied either by a variation of the magnetic field or by a variation of the area of the island, e.g., by means of a side gate. When the backscattering is measured as a function of one of these two parameters, an interference pattern is obtained, with a period corresponding to one flux quantum. In the case of the fractional quantized Hall state at $\nu = 1/3$, when the interference pattern is measured by varying the size of the island, its period corresponds to a change of three in the number of flux quanta enclosed by the island [10], reflecting the fact that the quasiparticles which tunnel across the point contacts carry a fractional charge of $e/3$.

We analyze this experiment for the $\nu = 5/2$ case, and find that it reflects the special character of this state in two ways. As in the Abelian fractional quantum Hall effect states, the period of the oscillations, when measured by varying the area of the island, reflects the $e/4$ charge of the elementary excitations. More interestingly, however, we consider the effect of $e/4$-charged quasiparticles localized statically within the island. We denote by n_{is} the number of these quasiparticles. We find that as a consequence of the non-Abelian character of the quasiparticles, the oscillations are suppressed when n_{is} is odd, and are revived when this number is even.

Following this analysis, we consider the limit of strong backscattering, where the island becomes a Coulomb-blockaded quantum dot. Measurements of the conductance through the dot allow in this limit an extraction of its

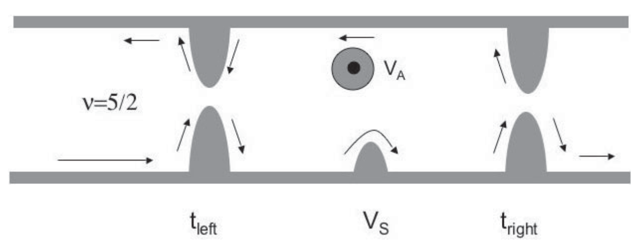

FIG. 1. Experimental setup for measuring the interference contribution to the backscattered current. Current flows along the lower edge, heading rightwards, and is backscattered by two quantum point contacts. The "island" is defined by the two quantum point contacts and the two edges. The antidot at the center of the island is coupled to an air-bridge gate that controls the number of $e/4$-charged quasiparticles that it localizes. A voltage V_S applied to a side gate varies the size of the island.

addition spectrum. We find similar sensitivity of the addition spectrum to the parity of $n_{\rm is}$.

In order to establish our results and study their consequences, we start by reviewing the basic theory of the non-Abelian $\nu = 5/2$ state [11]. The Moore-Read non-Abelian $\nu = 5/2$ quantum Hall state may be regarded as a p-wave superconductor of composite fermions. Precisely at $\nu = 5/2$, at zero temperature $T = 0$, this superconductor is free of topological defects. At a filling factor $\nu = \frac{5}{2} + \epsilon$, with $|\epsilon| \ll 1$, the superconductor is pierced by well-separated vortices. Then, each vortex i carries a zero-energy mode γ_i localized to its core. These modes may be written as Majorana fermions

$$\gamma_i = \int d\mathbf{r}[g_i(\mathbf{r})\psi(\mathbf{r}) + g_i^*(\mathbf{r})\psi^\dagger(\mathbf{r})], \tag{1}$$

where $\psi(\mathbf{r})$ annihilates a composite fermion at point $\mathbf{r}$. The function g_i is localized at the ith vortex core but has a phase that depends on the position of all other vortices. The Majorana operators satisfy $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$. As a consequence, when the vortices are pinned to their position, the ground state becomes degenerate. For $2n$ quasiparticles located at $\{R_1 \ldots R_{2n}\}$, the ground state subspace is of dimension 2^n, and it is spanned by the wave functions $\Psi_k\{R_1 \ldots R_{2n}\}$, in which the vortex positions are parameters, and the integer index $1 \le k \le 2^n$. A braiding of the vortex positions R_j's is accompanied by a unitary transformation acting within this subspace. These transformations may be expressed in terms of the Majorana operators γ_i. In particular, when a vortex i encircles vortex j, the unitary transformation is, up to a phase factor, $\gamma_i\gamma_j$ [12,13].

When a quasiparticle comes from $x = -\infty$ along the right moving edge, gets backscattered by one of the two point contacts, and moves back to $x = -\infty$ along the left-moving edge, it may end up encircling along its way the $n_{\rm is}$ quasiparticles localized within the bulk. In the presence of such localized quasiparticles, when the quasiparticle moving along the edge comes back to $x = -\infty$, it leaves the system in a ground state different from the one it started at. We denote the initial ground state as $|\text{initial}\rangle$, the unitary transformation applied on that state by the partial wave scattered by the right point contact as U_R, and the unitary transformation applied by the partial wave scattered at the left point contact as U_L. With this notation, the Ω-dependent part of the backscattered current becomes, to lowest order in the backscattering amplitudes,

$$2\,\mathrm{Re}[t_{\rm L}^* t_{\rm R} e^{2\pi i\Omega}\langle \text{initial}|U_{\rm L}^{-1}U_{\rm R}|\text{initial}\rangle]. \tag{2}$$

The definition of Ω involves an arbitrary additive constant, since $t_{\rm L}$, $t_{\rm R}$ are complex. The variation of Ω with the area A of the island satisfies,

$$\frac{\partial\Omega}{\partial A} = B/4\Phi_0 = n_0/2, \tag{3}$$

where B is the magnetic field, Φ_0 is the flux quantum, and n_0 is the density of electrons in the partly filled Landau level, as expected for quasiparticles with charge $e/4$. Denoting the Majorana mode of the interfering quasiparticle by γ_a, and those of the $n_{\rm is}$ quasiparticles localized at the island between the point contacts by γ_i, with $1 \le i \le n_{\rm is}$, we find that up to a possible phase that is independent of Ω and of the ground state at which the system is,

$$U_a \equiv U_l^{-1}U_r = \gamma_a^{n_{\rm is}}\Gamma, \tag{4}$$

with $\Gamma \equiv \prod_{i=1}^{n_{\rm is}} \gamma_i$. For our future discussion it is useful to note that the operator $U_a^2 = (-1)^{[3n_{\rm is}/2]}$, with $[3n_{\rm is}/2]$ denoting the integer part of $3n_{\rm is}/2$. Thus, U_a has two eigenvalues that differ by a minus sign (either ± 1 or $\pm i$).

It is in the expression (4) that the parity of $n_{\rm is}$ becomes crucial. Since for any even $n_{\rm is}$ we have $\gamma_a^{n_{\rm is}} = 1$, the product (4) of the two unitary transformations is independent of the Majorana mode of the incoming particle γ_a, and is the same for all incoming particles.

The two eigenvalues of Γ correspond to two interference patterns that are mutually shifted by 180°. If the initial ground state $|\text{initial}\rangle$ is an eigenstate of Γ then the phase of the interference pattern is determined by the corresponding eigenvalue. Furthermore, even if $|\text{initial}\rangle$ is not an eigenstate of Γ an interference pattern will be observed [14]. In that case, the electronic current driven from $x = -\infty$ and being backscattered from the two point contacts acts as a measuring device of Γ, as when enough quasiparticles flow through the system to ascertain the backscattering probability, the eigenvalue of Γ may be extracted from that probability. Thus, a measurement of the two terminal conductance of the system turns the superposition of different eigenvalues of Γ into a mixed state, and in effect collapses the system to one of the eigenvalues of Γ.

The physical distinction between the two subspaces that correspond to the two eigenvalues of Γ becomes clearer if one considers the limit of strong backscattering at the constrictions of Fig. 1, where the island becomes a closed system. Then, the subspace spanned by γ_i ($i = 1 \ldots n_{\rm is}$) is split into two subspaces of equal dimension $[2^{(n_{\rm is}/2)-1}]$ that correspond to a different parity of the total number of electrons in the closed island. The eigenvalues of the operator Γ distinguish between these two subspaces. In a closed system, this eigenvalue cannot be changed by operations that involve braiding between the $n_{\rm is}$ localized quasiparticles. Similarly, in the open system, one needs a quasiparticle exterior to the $n_{\rm is}$ localized ones to tunnel between the edges in order to change that eigenvalue (see Ref. [9] and the discussion at the end of this Letter).

The effect of the localized quasiparticles on the interference is very different when $n_{\rm is}$ is odd. Then, the unitary transformation U_a, Eq. (4), includes the Majorana operator of the interfering quasiparticle. For two different incoming quasiparticles a, b, the operators U_a, U_b do not commute. Rather, $[U_a, U_b] = (-1)^{[3n_{\rm is}/2]}\gamma_a\gamma_b$. Thus, for each incoming quasiparticle the interference term is multiplied by a different factor, and overall, the interference is dephased, and the backscattered current (2) becomes independent of Ω.

As shown above, the interference pattern to be observed in the setup of Fig. 1 depends crucially on the parity of $n_{\rm is}$. For the observation of such a dependence we need to increment $n_{\rm is}$ in a controlled fashion. Note that quasiparticles are introduced into the system by local deviations of ν from $5/2$. We consider three experimental knobs. The first is the side gate in Fig. 1. When the voltage on that side gate, V_S, is varied, the size of the island varies, but (ideally) the electron density is unchanged in the interior. Then Ω is varied, but no new quasiparticles are introduced. The second knob is the antidot near one edge of the island (see Fig. 1). We assume this antidot to be small enough such that its charging energy is larger than the temperature and therefore its charge is quantized in units of $e/4$ by the Coulomb blockade. Furthermore, we assume this charge to be variable by means of an air-bridged gate that couples to the antidot. When the voltage on that gate, V_A, is varied, the number of $e/4$ quasiparticles charging the antidot is varied, and thus so is also $n_{\rm is}$. The matrix element for quasiparticles to tunnel between the antidot and the edge of the island should be large enough so that tunneling can occur when the gate voltage is swept through the resonant condition but negligible when the antidot is off resonance. The third knob is the magnetic field B.

There are two experimental procedures to test the effect of the parity of $n_{\rm is}$. In the first procedure the magnetic field is kept fixed, and the backscattered current is measured as a function of the size of the island and the voltage V_A. We expect oscillations of the backscattering current as a function of V_S, and we expect the amplitude of these oscillations to vary discontinuously with V_A, changing periodically between zero and $O(t_{\rm L}^* t_{\rm R})$ as $n_{\rm is}$ is varied with V_A.

In the second procedure we turn off the antidot and vary $n_{\rm is}$ by varying B. If the density is kept fixed, the variation of B changes the filling factor uniformly within the island. For small deviations from $\nu = 5/2$, a set of localized quasiparticles will be introduced into the island, and $n_{\rm is}$ will vary with B. The positions and the precise values of B at which these quasiparticles will enter the island depend on the precise shape of the island and the disorder potential it encompasses. On average, a change in the magnetic field by one tenth of a flux quantum introduces one $e/4$-charged quasiparticle into the island, but fluctuations from that rate are to be expected. In any case, the backscattered current should again oscillate with V_S, and the amplitude of these oscillations should be turned on/off with the introduction of quasiparticles by the variation of B.

So far we have assumed that the parity of $n_{\rm is}$ is time independent throughout the experiment. For that assumption to be valid, the charge on the island should have typical fluctuations much smaller than $e/4$, or, if this condition is not realized, fluctuations whose characteristic time scale is much longer than that of the experiment. Assuming that the conductance of the island to the outside bulk, G, is frequency independent, and confining our attention to frequencies $\omega \ll T/\hbar$, the charge fluctuations on the island satisfy

$$\langle Q(t=0)Q(t)\rangle = 2CT\exp(-t/\tau), \tag{5}$$

where C is the capacitance of the island and $\tau = C/G$ is the relaxation time for the charge fluctuations. For thermal fluctuations of the charge to be much smaller than $e/4$ the capacitance of the island should satisfy $C \ll e^2/32T$. Typically, a dot of a 300 nm radius has a self capacitance of $\sim 3\times 10^{-16}$ F, corresponding here to a temperature of $\sim$200 mK [15]. The determination of the capacitance relevant to our case is rather subtle, however, since the bulk of the island, where the $n_{\rm is}$ quasiparticles are located, is electrostatically coupled to the edges, to the bulk outside of the island, and to various external gates.

Analogous phenomena may be observed in closed systems at $\nu = 5/2$. We consider a closed island with $n_{\rm is}$ pinned quasiparticles in its bulk, and study the way the energy of the island varies when its area is varied by the application of a voltage V_S to a side gate. For simplicity, we first disregard the two filled Landau levels, and incorporate their effect later. We assume a very weak coupling of the island to an electron reservoir, such that as the area is varied, the number of electrons in the island varies as well, but for any fixed V_S, this number is fixed to an integer.

Since the $\nu = 5/2$ state is a superconductor of composite fermions, whose number equals the number of electrons n_e, one may expect even-odd oscillations of the energy as a function of n_e, reflecting the difference between a fully paired ground state of a superconducting island and one in which one electron is unpaired [16]. The p-wave superconductor of composite fermions that we discuss here is rather unconventional in having a subgap excitation branch near the edge [11]. When the area of the island is increased and electrons are added, the unpaired electrons, if any, occupy the lowest state of that branch. And it is in the energy of that state, which we denote by δ_0, that the parity of $n_{\rm is}$ has an effect: When $n_{\rm is}$ is odd, $\delta_0 = 0$, and the dependence of the energy on the number of electrons does not show any even-odd effect. In contrast, when $n_{\rm is}$ is even, δ_0 is small (inversely proportional to the perimeter of the island), *but nonzero* [11]. In this case, as the number of electrons in the island is increased the energy cost for adding an electron depends on whether the added electron is paired or unpaired. For a closed system, then, the even-odd effect in the energy cost associated with changing the electron number by one is turned off when $n_{\rm is}$ is odd and turned on when $n_{\rm is}$ is even.

A practical way of measuring this even-odd effect may use the system in Fig. 1 in the limit of strong backscattering, at which a quantum dot of $\nu = 5/2$ is formed between the point contacts, and the side gate is used to vary the dot's area. If the two terminal conductance of the dot is measured as a function of the gate voltage V_S, the well-known series of conductance peaks is to be expected, associated with values where the number of electrons on the dot changes by one. The voltage separation between these peaks measures the energy cost involved in adding an extra electron, and is the quantity that should reflect the parity of $n_{\rm is}$. In the case where $n_{\rm is}$ is odd, the average spacing between peaks corresponds to an area change $\delta A = 1/n_0$. In the case where it is even, there will be even-odd fluctuations about this average, so that the true period becomes $2/n_0$. The parity of $n_{\rm is}$, in turn, may be varied by a variation of the magnetic field. Again, on average $n_{\rm is}$ is varied by one when the flux through the dot is varied by $1/10$ of a flux quantum.

For a closed island, a change in V_S affects also the occupation of the two filled Landau levels, and introduces additional Coulomb blockade peaks associated with this occupation. The periodicity of these peaks corresponds to $\delta A = 1/n_0$, and therefore does not eliminate the distinction between odd and even $n_{\rm is}$.

The limits of weak and strong backscattering may be compared through a Fourier decomposition of the conductance as a function of the area of the island. We write $G = \sum_m g_m e^{2\pi i m\Omega}$, where Ω is defined by (3). Terms with m odd should be absent when $n_{\rm is}$ is odd and present when $n_{\rm is}$ is even. With weak backscattering successive terms get smaller by the small factor of $t_{\rm L}^* t_{\rm R}$, while in the limit of strong backscattering no such small parameter exists.

Since for a closed system the number of electrons is quantized to an integer, thermal fluctuations of the charge on the island must be much smaller than the electron charge, and thus the capacitance of the dot should satisfy $C \ll e^2/2T$. The requirement $T \ll \delta_0$ leads to a similar condition, if we use the estimate $\delta_0 = \kappa e^2/2C$, where κ is a parameter of order unity, determined mostly by the smoothness of the edge [11,15,18]).

The experiments we suggest here are simpler than the experiment suggested by Das Sarma *et al.* [9], but the goal we address is less ambitious. In Ref. [9] $n_{\rm is} = 2$, and at the crucial part of the experiment the eigenvalue of Γ is changed and the interference pattern is shifted by 180°. In practice, it is probably very difficult to tune $n_{\rm is}$ precisely to two, due to the unavoidable abundance of localized quasiparticles resulting from density nonuniformities. We showed that when $n_{\rm is}$ is odd, no interference is to be observed. We further examine the case when $n_{\rm is}$ is an even number different from two. In that case a measurement of the backscattered current as a function of the side gate voltage V_S collapses the system into a ground state with one of the two possible eigenvalues of Γ. In order to change that eigenvalue we need to apply a unitary transformation $\tilde{\Gamma}$ that does not commute with Γ. It is easy to see that if the even-numbered $n_{\rm is}$ localized quasiparticles are separated into two odd-numbered groups of quasiparticles, and a quasiparticle from the edge encircles the quasiparticles of one of these two groups (say group number 1), the resulting unitary transformation $\gamma_a \prod_i^{(1)} \gamma_i$ (where $\prod_i^{(1)} \gamma_i$ indicates a product over all Majorana operators of the quasiparticles of group number 1) changes the eigenvalue of Γ. As suggested by Das Sarma *et al.* [9], this transformation may be applied by a single quasiparticle tunneling between the edges through another quantum point contact, situated between the left and right ones. This point contact divides the island into two parts. The present analysis reveals, then, that the Das Sarma *et al.*, procedure would indeed shift the interference pattern by 180° only if each of these two parts includes an odd number of quasiparticles.

To summarize, in this Letter we propose several experiments that probe the non-Abelian character of the $\nu = 5/2$ quantum Hall state, both through transport measurements in an open system and through thermodynamic measurements in a closed system.

We thank A. Kitaev and C. M. Marcus for instructive discussions. We acknowledge support from the US-Israel Binational Science Foundation, the Minerva Foundation (A. S.), and NSF Grant DMR 02-33773 (B. I. H.).

[1] G. Moore and N. Read, Nucl. Phys. **B360**, 362 (1991).
[2] N. Read, Physica B (Amsterdam) **298**, 121 (2001); N. Read and E. Rezayi, Phys. Rev. B **59**, 8084 (1999).
[3] R. Morf and N. d'Ambrumenil, Phys. Rev. B **68**, 113309 (2003).
[4] J. S. Xia *et al.*, J. Low Temp. Phys. **134**, 579 (2004).
[5] N. R. Cooper *et al.*, Phys. Rev. Lett. **87**, 120405 (2001).
[6] J. W. Reijnders *et al.*, Phys. Rev. A **69**, 023612 (2004).
[7] A. Y. Kitaev, Ann. Phys. (Berlin) **303**, 2 (2003).
[8] E. Fradkin *et al.*, Nucl. Phys. **B516**, 704 (1998).
[9] S. Das Sarma, M. Freedman, C. Nayak, Phys. Rev. Lett. **94**, 166802 (2005).
[10] C. D. C. Chamon *et al.*, Phys. Rev. B **55**, 2331 (1997).
[11] N. Read, D. Green, Phys. Rev. B **61**, 10 267 (2000).
[12] C. Nayak and F. Wilczek, Nucl. Phys. **B479**, 529 (1996).
[13] D. A. Ivanov, Phys. Rev. Lett. **86**, 268 (2001); A. Stern, F. von Oppen, and E. Mariani, Phys. Rev. B **70**, 205338 (2004).
[14] B. J. Overbosch and F. A. Bais, Phys. Rev. A **64**, 062107(2001).
[15] See, e.g., L. P. Kouwenhoven *et al.*, Z. Phys. B **85**, 367 (1991).
[16] M. T. Tuominen *et al.*, Phys. Rev. Lett. **69**, 1997 (1992); Phys. Rev. B **47**, 11 599 (1993).
[17] P. Lafarge *et al.*, Phys. Rev. Lett. **70**, 994 (1993); Nature (London) **365**, 422 (1993).
[18] G. Ernst *et al.*, Phys. Rev. Lett. **79**, 3748 (1997).