Modulational Instability in Crossing Sea States: A Possible Mechanism for the Formation of Freak Waves

M. Onorato, A. R. Osborne, and M. Serio
Dipartimento di Fisica Generale, Università di Torino, Via Pietro Giuria 1, 10125 Torino, Italy
(Received 29 July 2005; published 10 January 2006)

Here we consider a simple weakly nonlinear model that describes the interaction of two-wave systems in deep water with two different directions of propagation. Under the hypothesis that both sea systems are narrow banded, we derive from the Zakharov equation two coupled nonlinear Schrödinger equations. Given a single unstable plane wave, here we show that the introduction of a second plane wave, propagating in a different direction, can result in an increase of the instability growth rates and enlargement of the instability region. We discuss these results in the context of the formation of rogue waves.

DOI: 10.1103/PhysRevLett.96.014503 PACS numbers: 47.35.−i, 92.10.Hm

Recently wave systems characterized by two different spectral peaks, also known as crossing sea states, have become of particular interest in the community of ocean waves. A study carried out by Toffoli *et al.* [1], based on data collected from 1995 to 1999 by Lloyd's Marine Information Service, has revealed that a large percentage of ship accidents due to bad weather conditions have occurred in crossing sea states. Lehner *et al.* [2], analyzing synthetic aperture radar (SAR) images, have pointed out that the famous Draupner time series, in which a wave of 26 meters height was measured, has also been recorded in conditions of crossing sea states. This condition is quite common in the ocean and occurs when a wind sea and a swell coexist (a wind sea is a wave system that is being forced by the local wind field; a swell is a system of waves, generated elsewhere, that have moved out of the generating area or are no longer affected by the local wind).

A complete understanding of such "two-phase" wave trains is far from being clear and the possible formation of extreme events resulting from a crossing sea state has not yet been investigated. Apart from a trivial linear superposition of two-wave systems (see [2] or [3]), which can eventually generate a large amplitude wave (simple interference mechanism), the role of weakly nonlinear interactions in the formation of extreme waves in crossing sea states has not received attention. Some results in shallow water using the Kadomtsev-Petviashvili equation have been described in [4], where the interaction between two solitons propagating in different directions has been considered as a possible model for extreme waves. Here we consider the case of infinite water depth and study the modulational instabilities that can arise in the dynamics of two-wave trains that propagate in different directions. This is a natural extension of previous works ([5–8]) where the modulational instability (Benjamin-Feir instability) was considered as a possible mechanism for the formation of extreme waves (recent work has shown that this mechanism is particularly relevant for long crested waves [9–11]). In these papers the basic equation that has been considered as a starting point for theoretical considerations is the nonlinear Schrödinger (NLS). This equation can be derived from the inviscid, irrotational primitive equation of motions in a weakly nonlinear regime, under the hypothesis that wave energy is basically concentrated in a single wave number (the carrier wave). A well-known result is that the plane wave solution of NLS can be unstable to side-band, small amplitude perturbations [12]. According to the NLS equation, the evolution of an unstable wave group generates a single wave that can reach up to 3 times the amplitude of the initial carrier wave. Because of the hypothesis under which the NLS equation is derived (narrow banded approximation), it is obvious that it cannot describe the evolution of two-wave systems characterized by well-separated wave numbers. In order to properly approach the problem of crossing sea states, in this Letter we derive two coupled nonlinear Schrödinger equations (CNLS), each describing the evolution of a single spectral peak; we then study the stability properties of the systems.

Before entering into the discussion, it should be mentioned here that the CNLS equations have been discussed in different fields of physics. For example, focusing CNLS equations have been derived in [13,14] and in general they can be derived for nonlinear media for two waves with different polarizations. For particular values of the coefficients in the CNLS equations, it has also been shown that the system is integrable [15] (see also [16]). Concerning water waves, the CNLS equations have been discussed in many different papers [17–26]. It should also be mentioned that most of these papers either discuss the one-dimensional problem of standing and copropagating waves or, in the two-dimensional case, do not report the coefficients of the terms in the CNLS equations which are fundamental for a detailed stability analysis (in [26] coefficients for the CNLS equations are reported but no stability analysis has been performed). To the knowledge of the authors no approach to the study of extreme waves has been attempted before using the CNLS equations.

Here our starting point is the Zakharov equation in two spatial dimensions ($2D + 1$):

$$\frac{\partial a_1}{\partial t} + i\omega_1 a_1 = -i \int T_{1,2,3,4} a_2^* a_3 a_4 \delta_{1,2}^{3,4} d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4, \quad (1)$$

where $\delta_{1,2}^{3,4} = \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)$, $\omega = \sqrt{g|\mathbf{k}|}$. The integral is six dimensional on $-\infty$ to $+\infty$. $T_{1,2,3,4} = T(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$ is the coupling coefficient whose analytical form can be found in [27]. The complex amplitude $a_i = a(\mathbf{k}_i, t)$ is related to the surface elevation $\eta(\mathbf{x}, t)$ to the leading order as follows [dependence on time of $a(\mathbf{k}_i, t)$ will be omitted for brevity]:

$$\eta(\mathbf{x}, t) = \int_{-\infty}^{+\infty} \sqrt{\frac{|\mathbf{k}|}{2\omega(|\mathbf{k}|)}} [a(\mathbf{k}) + a^*(-\mathbf{k})] e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k}. \quad (2)$$

We consider the case of energy concentrated mainly around two-wave numbers: $\mathbf{k} = \mathbf{k}^{(a)}$ and $\mathbf{k} = \mathbf{k}^{(b)}$. Consequently we write the complex amplitude $a(\mathbf{k})$ as:

$$a(\mathbf{k}) = A(\mathbf{k} - \mathbf{k}^{(a)}) e^{-i\omega^{(a)}t} + B(\mathbf{k} - \mathbf{k}^{(b)}) e^{-i\omega^{(b)}t} \quad (3)$$

with $\omega^{(a)} = \sqrt{g|\mathbf{k}^{(a)}|}$ and $\omega^{(b)} = \sqrt{g|\mathbf{k}^{(b)}|}$. Using (3), under the hypothesis that $\mathbf{k}^{(a)} \neq \mathbf{k}^{(b)}$, it is possible to write Eq. (1) as two coupled equations. We concentrate our analysis on the particular case of $\mathbf{k}^{(a)} = (k, l)$ and $\mathbf{k}^{(b)} = (k, -l)$ and consider both energy distributions as quasimonochromatic. In this case, in each equation, $\omega(|\mathbf{k}|)$, can be Taylor expanded around the two dominant wave numbers. In order to balance nonlinearity and dispersion, in the linear part of the equation the expansion is performed up to second order while in the nonlinear part, only the leading order term is considered. The methodology is standard and has been used in the past to derive the single NLS equation (and higher order corrections) starting from the Zakharov equation (see for example [25]). Scaling the variables A and B with $\sqrt{2|\mathbf{k}|/\omega(|\mathbf{k}|)}$ so that they have the dimension of a surface elevation, we get the following CNLS:

$$\frac{\partial A}{\partial t} + C_x \frac{\partial A}{\partial x} + C_y \frac{\partial A}{\partial y} - i\alpha \frac{\partial^2 A}{\partial x^2} - i\beta \frac{\partial^2 A}{\partial y^2} + i\gamma \frac{\partial^2 A}{\partial x \partial y} + i(\xi|A|^2 + 2\zeta|B|^2)A = 0 \quad (4)$$

$$\frac{\partial B}{\partial t} + C_x \frac{\partial B}{\partial x} - C_y \frac{\partial B}{\partial y} - i\alpha \frac{\partial^2 B}{\partial x^2} - i\beta \frac{\partial^2 B}{\partial y^2} - \gamma \frac{\partial^2 B}{\partial x \partial y} + i(\xi|B|^2 + 2\zeta|A|^2)B = 0. \quad (5)$$

The coefficients for the linear terms are

$$C_x = \frac{\omega(\kappa)}{2\kappa^2} k, \qquad C_y = \frac{\omega(\kappa)}{2\kappa^2} l,$$
$$\alpha = \frac{\omega(\kappa)}{8\kappa^4}(2l^2 - k^2), \qquad \beta = \frac{\omega(\kappa)}{8\kappa^4}(2k^2 - l^2), \quad (6)$$
$$\gamma = -\frac{3\omega(\kappa)}{4\kappa^4} lk,$$

and for the nonlinear terms are:

$$\xi = \frac{\omega(\kappa)}{2\kappa} T_{a,a,a,a} = \frac{\omega(\kappa)}{2\kappa} T_{b,b,b,b} = \frac{1}{2}\omega(\kappa)\kappa^2 \zeta$$
$$= \frac{\omega(\kappa)}{2\kappa} T_{a,b,b,a} = \frac{\omega(\kappa)}{2\kappa} T_{b,a,a,b}$$
$$= \frac{\omega(\kappa)}{2\kappa} \frac{k^5 - k^3 l^2 - 3kl^4 - 2k^4\kappa + 2k^2 l^2 \kappa + 2l^4 \kappa}{(k - 2\kappa)\kappa,}, \quad (7)$$

where $\kappa = \sqrt{k^2 + l^2}$. Similar equations have been recently derived in [26] starting from the Euler equations and using the method of the multiple scales (our coefficient ζ is slightly different from the one reported in [26]). We now consider the stability analysis of Eqs. (4) and (5) for perturbations along the k_x axes. For these perturbations the stability analysis can be performed directly on the following equations:

$$\frac{\partial A}{\partial t} - i\alpha \frac{\partial^2 A}{\partial x^2} + i(\xi|A|^2 + 2\zeta|B|^2)A = 0 \quad (8)$$

$$\frac{\partial B}{\partial t} - i\alpha \frac{\partial^2 B}{\partial x^2} + i(\xi|B|^2 + 2\zeta|A|^2)B = 0. \quad (9)$$

This system is quite general and, with different coefficients, can be obtained in the case of two arbitrary quasimonochromatic wave systems by making the hypothesis that variations take place only in the direction perpendicular to the one determined by $\mathbf{k}^{(a)} - \mathbf{k}^{(b)}$ (see [19]). Note that to obtain (8) and (9) we have performed a Galilean transformation of the form of $x' = x - C_x t$. The possibility of removing the term containing the group velocity in both equations makes the system (8) and (9) as "nice" as the NLS equation in the sense that, if written in nondimensional form, it is possible to find a time scale (and a space scale) for which the small parameter used in the asymptotic expansion to derive these equations can be formally removed. In general this property is not shared by (4) and (5) because each wave system travels with its own group velocity. It is also interesting to mention that the coefficient α changes sign (from negative to positive) when $\theta = \arctan(l/k) > 35.264°$; the system of equations moves from focusing to defocusing CNLS; therefore, we expect that this transition may affect the stability properties of the solutions.

We now consider the following plane wave solution of CNLS:

$$A = A_0(1 + a)e^{-i(\omega t + \phi_a)}, \qquad B = B_0(1 + b)e^{-i(\omega t + \phi_b)}, \quad (10)$$

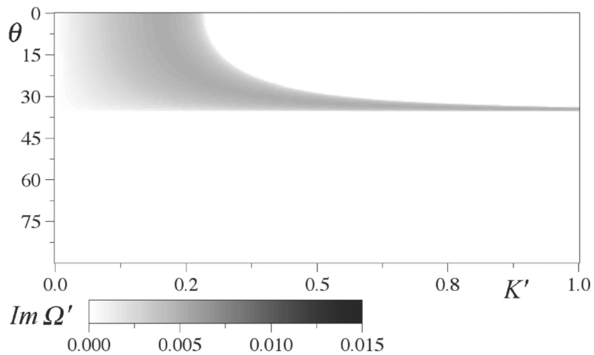

FIG. 1. Instability diagram for $B_0 = 0$ as a function of angle θ and perturbation wave number. See text for details.

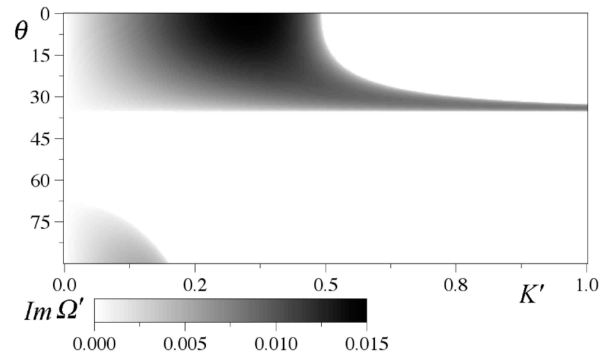

FIG. 2. Instability diagram for $B_0 = A_0$ as a function of angle θ and perturbation wave number. See text for details.

where a, b, ϕ_a, ϕ_b are small perturbations in amplitude and phase of the plane wave solutions. We substitute (10) in (8) and (9), then linearize the resulting equations and use the normal mode approach, with K and Ω respectively the wave number and the angular frequency of the perturbation, to obtain the following dispersion relation:

$$\Omega = \pm\sqrt{\alpha K^2[(\xi(A_0^2 + B_0^2) + \alpha K^2) \pm \sqrt{\xi^2(A_0^2 - B_0^2)^2 + 16\zeta^2 A_0^2 B_0^2}]}. \tag{11}$$

The first terms under the square root correspond, in the case of B_0, to the standard Benjamin-Feir instability, while the last term under the second square root is the result of the presence of the second carrier wave in the system. In Figs. 1 and 2 we show the imaginary part of Ω' for two different cases as a function of the wave number of the perturbation K' and the angle $\theta = \arctan(l/k)$. The growth rate $\mathrm{Im}[\Omega']$ and wave number K' reported in the figures are nondimensional ($\Omega' = \Omega/\sqrt{g\kappa}$ and $K' = K/\kappa$). Growth rates are presented in gray scale: white regions correspond to zero growth rates (stability) and the darker regions to instability. In Fig. 1 the growth rate for $\epsilon_A = A_0\kappa = 0.1$ and $B_0 = 0$ is considered. Note that we are looking at the evolution of the perturbation along the k_x axes and not along the direction of propagation of the wave; when $B_0 = 0$, our resulting instability curve corresponds to slices along the k_x axes of the instability diagram of the $2D + 1$ NLS equation for different angles of propagation (note that the carrier wave does not lie on the k_x axes). As expected, Fig. 1 shows that for $\theta > 35.264°$ any perturbation is stable. The unstable region that extends to high wave numbers around $\theta = 35°$ corresponds to the unbounded region of instability of the $2D + 1$ NLS equation, which is well known to be unphysical. Figure 2 corresponds to the case of two carrier waves with the same steepness $A_0\kappa = B_0\kappa = 0.1$. With respect to Fig. 1 the growth rate has increased as can be seen from darker regions in the instability plot. Moreover, Fig. 2 shows an unstable region also for large angles ($68.02° < \theta \leq 90°$): two waves traveling in almost opposite directions are unstable to perturbations perpendicular to their directions of propagation (see, for example, [28]).

In order to appreciate more the differences in the growth rates of the two cases, we show in Fig. 3 a slice of Figs. 1 and of Fig. 2 for $\theta = 15.3°$. The larger growth rates for the crossing sea state is evident from the figure. In Fig. 4 we show different slices of Fig. 2 corresponding to different angles. As can be seen from the figure the growth rate increases as the angle becomes smaller; this is consistent with results in [28,29]. The maximum growth rate is achieved when two waves propagate almost in the same direction; in this case the two-wave system, each of steepness ϵ, tends to a single system of steepness 2ϵ.

In order to check the consistency of the growth rates obtained, we have performed direct numerical simulations of the CNLS equations in (8) and (9). We have used a standard pseudospectral numerical method in which the linear part is solved exactly in Fourier space and the nonlinear terms are solved in physical space. In Fig. 5 we show the evolution in time of the mode $K' = 0.226$ for $\theta = 15.3°$ respectively for the case of $B_0 = 0$ and $\kappa A_0 = \kappa B_0 = 0.1$. We have selected the evolution of the mode at $K' = 0.226$ because it corresponds to the most unstable mode for $B_0 = 0$. Exponential curves with the growth rates predicted by Eq. (11) are also shown in the figure: the slopes of the curves obtained from numerical simulations

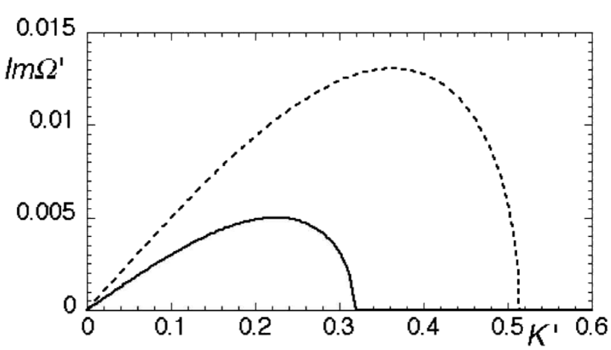

FIG. 3. Nondimensional growth rates for longitudinal perturbation of a single carrier wave (solid line) and of two carrier waves (dotted line).

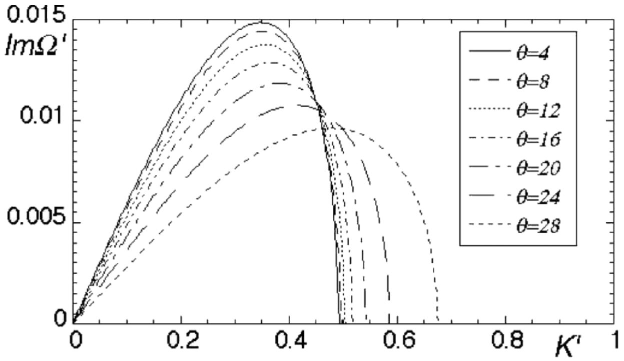

FIG. 4. Growth rates as a function of perturbation wave number for different angles θ; in the legend angles are expressed in degrees.

are in very good agreement with theoretical curves. These results give us some confidence that the growth rates are calculated correctly. The figure shows also that, as expected, after some period of time the nonlinearity becomes important and the perturbed modes do not grow indefinitely.

To conclude we have studied theoretically the influence of a second wave system propagating in a different direction to the first. We have derived a simple model for the interaction of two-wave systems propagating in different wave directions. Results show that the instability region and the growth rates are larger when two-wave systems are considered. It is clear that this is a very idealized case and definitely more realistic conditions of random spectra should be considered. Nevertheless our goal was to investigate a basic physical mechanism that can in principle be responsible for the formation of extreme events in crossing sea states. Comparison with direct numerical simulations of the primitive equations should be performed in order to verify the relevance of the present results in more realistic conditions.

We thank A. Toffoli for discussions on crossing sea states, P. Janssen for comments during the early stages of the work, and D. Fuhrman for pointing out a number of references on "short crested" waves. The work has been supported by MIUR (PRIN project) and by ONR.

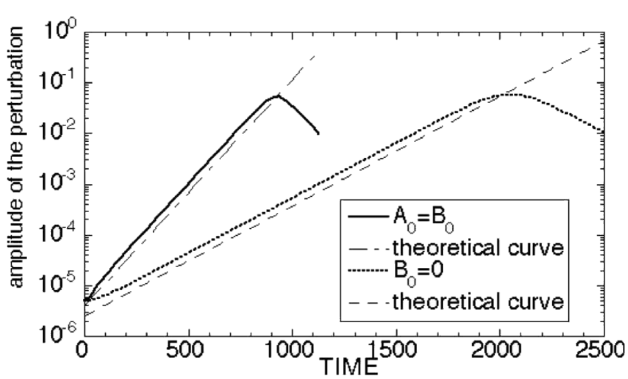

FIG. 5. Evolution in time of mode $K' = 0.226$ for $\theta = 15.3°$ for the case of $B_0 = 0$ and $A_0\kappa = B_0\kappa = 0.1$. Time has been normalized with $\omega(\kappa)$. The exponential curves are of the form of $e^{0.01028t}$ and $e^{0.005t}$.

[1] A. Toffoli, J. M. Lefevre, J. Monbaliu, and E. Bitner-Gregersen, *The Proceedings of the 14th International Offshore and Polar Engineering Conference (ISOPE), Toulon, France, May 23–28, 2004* (International Society of Offshore and Polar Engineers, Cupertino, CA, 2004).
[2] S. Lehener, H. Günther, and W. Rosenthal, Fifth International Symposium on Ocean Wave Measurement and Analysis, WAVES 2005, July 3–7, 2005, Madrid, Spain (unpublished).
[3] M. Donelan and A. K. Magnusson, *Proceedings of the Hawaiian Winter Workshop, University of Hawaii at Manoa, January 24–28, 2005*, edited by P. Muller and D. Henderson (SOEST Publications, Honolulu, 2005).
[4] P. Peterson, T. Soomere, J. Engelbrecht, and E. van Groesen, Nonlin. Proc. Geophys. **10**, 503 (2003).
[5] M. Onorato, A. R. Osborne, M. Serio, and S. Bertone, Phys. Rev. Lett. **86**, 5831 (2001).
[6] A. R. Osborne, M. Onorato, and M. Serio, Phys. Lett. A **275**, 386 (2000).
[7] K. B. Dysthe and K. Trulsen, Phys. Scr. **T82**, 48 (1999).
[8] P. A. E. M. Janssen, J. Phys. Oceanogr. **33**, 863 (2003).
[9] M. Onorato, A. R. Osborne, M. Serio, L. Cavaleri, C. Brandini, and C. T. Stansberg, Phys. Rev. E **70**, 067302 (2004).
[10] M. Onorato, A. R. Osborne, M. Serio, and L. Cavaleri, Phys. Fluids **17**, 078101 (2005).
[11] H. Socquet-Juglard, K. Dysthge, K. Trulsen, H. E. Krogstad, and J. Liu, J. Fluid Mech. **542**, 195 (2005).
[12] V. Zakharov, J. Appl. Mech. Tech. Phys. **9**, 190 (1968).
[13] A. L. Berkhoer and V. E. Zakharov, Sov. Phys. JETP **31**, 486 (1970).
[14] C. R. Menyuk, IEEE J. Quantum Electron. **QE-23**, 174 (1987).
[15] S. V. Manakov, Sov. Phys. JETP **38**, 248 (1974).
[16] M. G. Forest, D. W. McLaughlin, D. J. Muraki, and O. C. Wright, J. Nonlinear Sci. **10**, 291 (2000).
[17] D. J. Benney and A. C. Newell, J. Math. Phys. (N.Y.) **46**, 133 (1967).
[18] D. J. Benney and G. J. Roskes, Stud. Appl. Math. **48**, 377 (1969).
[19] G. J. Roskes, Stud. Appl. Math. **55**, 231 (1976).
[20] T. J. Bridges and F. E. Laine-Pearson, Stud. Appl. Math. **107**, 137 (2001).
[21] M. Okamura, J. Phys. Soc. Jpn. **53**, 3788 (1984).
[22] S. Badulin (private communication).
[23] M. Onorato, D. Ambrosi, A. R. Osborne, and M. Serio, Phys. Fluids **15**, 3871 (2003).
[24] R. D. Pierce and E. Knobloch, Phys. Fluids **6**, 1177 (1994).
[25] M. Stiassnie, Wave Motion **6**, 431 (1984).
[26] J. L. Hammack, D. M. Henderson, and H. Segur, J. Fluid Mech. **532**, 1 (2005).
[27] V. P. Krasitskii, J. Fluid Mech. **272**, 1 (1994).
[28] M. Ioualalen and C. Kharif, J. Fluid Mech. **262**, 265 (1994).
[29] S. I. Badulin, V. I. Shrira, C. Kharif, and M. Ioualalen, J. Fluid Mech. **303**, 297 (1995).