Heteroclinic Synchronization: Ultrasubharmonic Locking

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According to the traditional view of synchronization, a weak periodic input is able to lock a nonlinear oscillator at a frequency close to that of the input (1:1 zone). If the forcing increases, it is possible to achieve synchronization at subharmonic bands also. Using a competitive dynamical system we show the inverse phenomenon: with a weak signal the 1:1 zone is narrow, but the synchronization of ultrasubharmonics is dominant. In the system's phase space, there exists a heteroclinic contour in the autonomous regime, which is the image of sequential dynamics. Under the action of a weak periodic forcing, in the vicinity of the contour a stable limit cycle with long period appears. This results in the locking of very low-frequency oscillations with the finite frequency of the forcing. We hypothesize that this phenomenon can be the origin for the synchronization of slow and fast brain rhythms.

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Introduction.—The synchronization of periodic dynamics is usually associated with a "devil's staircase" whose main step is the 1:1 zone of synchronization and with Arnold tongues that become wider as the forcing is increased [1]. However, the synchronization of low-frequency oscillators by a high-frequency forcing, when such synchronization exists, is characterized by quite different features. Using as an example the long period oscillations that are observed in systems with winnerless competition (WLC), we analyze below the ultrasubharmonic synchronization phenomenon. Competition without a winner based on heteroclinic cycles is a robust oscillatory behavior in diverse multiagent systems, such as biological communities (the simplest example is the competition between three species [2]), structured hydrodynamic flows (e.g., competition between different convective patterns of rotating layers of fluid heated from below [3]), Earth magnetic field dynamics [4,5], cellular flame dynamics in porous media [6,7], mode interaction in counterrotating von Karman swirling flow [8], multimode lasers [9,10], etc. In this Letter, we use as an example the competitive activity of neural circuits and their synchronization by weak high-frequency input.

Competitive neural dynamics are responsible for the generation of low-frequency oscillations (~5 Hz) in the visual cortex of monkeys responding to visual stimuli [11,12]. WLC is presumably the dynamics observed in the antennal lobe, a first relay station of the insect olfactory system [13–15]. Chaotic WLC sensory dynamics has also been proposed as the origin of the irregular hunting behavior of the marine mollusk *Clione* [16–18]. The key control parameters for competitive neural circuits with inhibition are the degree of nonsymmetry of the inhibitory connections and the strength of inhibition between competing neuronal pools. In recurrent circuits with nonsymmetric connections, an increase of the strength of inhibition leads to a change of system dynamics from the coexisting activity of many pools to a "winner take all" regime or to a sequential switching of activities among different pools, i.e., WLC behavior [13].

Model.—Lotka-Volterra–type equations are convenient mathematical models to implement robust heteroclinic cycles [19,20]. In particular, these models are suitable for the description of synchronization phenomena in competitive networks. The mathematical image of WLC dynamics in the phase space of such models is a closed heteroclinic contour that consists of saddle fixed points and one-dimensional separatrices connecting them. In reality, because of the presence of a small amount of noise or intrinsic fluctuations, it can be not a strict heteroclinic contour but a long period limit cycle in its vicinity, which corresponds to the rhythmic activity of the network. Thus, we consider the model

$$\dot{a}_i = a_i\left[1 - \left(a_i + \sum_{j\neq i}^{N} \rho_{ij} a_j\right)\right] + \xi(t) + \gamma\varphi_i(\omega_f t, a_i). \tag{1}$$

Here $a_i(t) \geq 0$ represents the rate of the neural activity of the ith neuronal pool, ρ_{ij} is an inhibitory connection matrix, where $0 < \rho_{ij} < 1 < \rho_{ji}$, $\rho_{ii} = 1$, $\varphi_i(\omega t, a_i) \geq 0$ is a periodic function with period $T = 2\pi/\omega$, $\xi(t)$ is a Gaussian noise with zero correlation $\langle \xi(t)\xi(t')\rangle = \sigma\delta(t - t')$, and $0 < \gamma \ll 1$. These nonsymmetrical connections satisfy the necessary condition for the stability of the heteroclinc cycle. For $N = 3$, if $\nu_1\nu_2\nu_3 > 1$, where $\nu_i = (\rho_{i,i+2} - 1)/(1 - \rho_{i,i+1})$ (i a cyclic index), the heteroclinic contour or limit cycle in its vicinity is a global attractor [21]. The period T of the noise induced cycles depends on the level of noise as [22]:

$$T \sim |\ln\sigma|/\lambda_u, \tag{2}$$

where λ_u is a positive Lyapunov exponent that characterizes the one-dimensional unstable separatrices of the saddle points (for simplicity, we suppose that they are identical).

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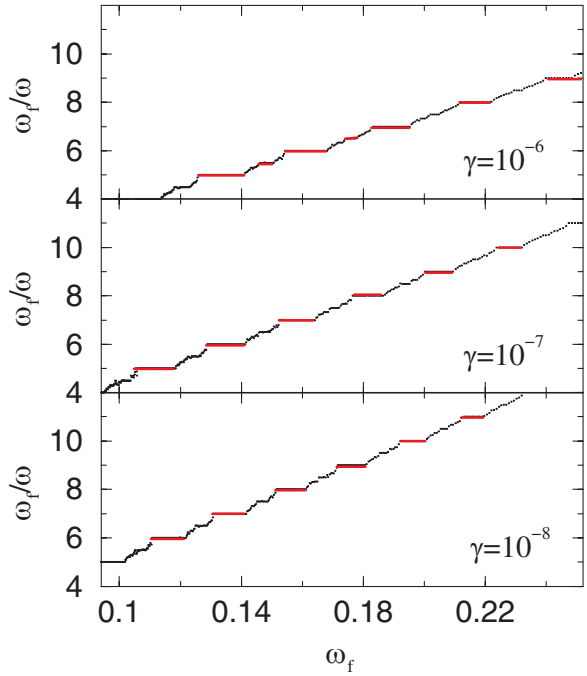

FIG. 1 (color online). Devil's staircase in a model described by (1) with $N = 3$. Here ω is the observed frequency of oscillation and ω_f is the forcing frequency. The parameter values are $\rho_{ii} = 1.0$, with $i = 1, \ldots, 3$, $\rho_{12} = \rho_{23} = \rho_{31} = 1.25$, and $\rho_{13} = \rho_{21} = \rho_{32} = 0.8$. In addition, $\varphi_1(\omega_f t, a_1) = (1 - a_1) \times (\sin(\omega_f t) + 1)$, the first multiplier guarantees that the system dynamics is in the subspace $a_i \geq 0$, and $\varphi_2(t) = \varphi_3(t) = 0$. The noise level in this example is $\sigma = 0$.

Ultrasubharmonic synchronization.—In Fig. 1, we show the synchronization bands for different levels of strength of the forcing γ in a system of three units described by (1). A very small strength of the stimulus is able to induce ultrasubharmonic synchronization that remains in the presence of noise. Figure 2 shows that the width of the synchronization bands are squeezed for $\gamma \to 0$ that corresponds to the heteroclinic orbit. This figure also shows that the synchronization bands of the system, as a function of the forcing frequency, have a fairly smooth dependence on the forcing, which indicates that the system near a heteroclinic trajectory has a good degree of flexibility to lock in a wide range of ultrasubharmonics, i.e., multistability.

Origin of the flexibility.—To explain the origin of the flexibility of the system, we introduced increasing levels of additive noise σ in system (1). Then we estimated the dependence of the oscillation period T on the vicinity of the heteroclinic orbit as a function of σ. A simple calculation indicated that most of the time is taken by the unstable manifold. Therefore, the mean period of the oscillations should be described by (2). In Fig. 3, we show the result of the simulation of the previously described system with a clear logarithmic dependence on the noise level. This feature exposes the ability of the system to lock in such a variety of frequency bands in the proximity of the heteroclinic orbit.

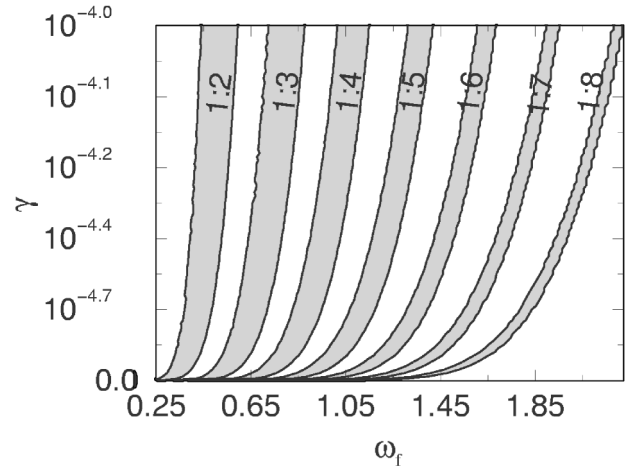

FIG. 2. Bands of synchronization in system (1) as a function of the strength of the forcing γ for some representative subharmonics. The parameter values are the same as in Fig. 1.

As an illustration of the complexity of the bifurcations near the synchronization bands, we show in Fig. 4 the period as a function of the forcing frequency. The region in between synchronized regimes displays chaos. If, instead of plotting the period changes, the time series are plotted, chaos is not apparent.

Map.—To generalize the framework of ultrasubharmonic synchronization, we have built a heteroclinic circle map as follows:

$$\Psi_{n+1} = G(\Psi_n) + \beta F(\Psi_n), \tag{3}$$

where the function $G(\Psi_n)$ describes the motion along the heteroclinic cycle in the phase space of the autonomous system when $\beta = 0$. $F(\Psi_n)$ is the forcing function with frequency ω_f. Figure 5 shows the bands of synchronization of a map that reproduces the main features of the ultrasubharmonic synchronization observed in the previously studied system (1). This map is made of three exponentials as follows:

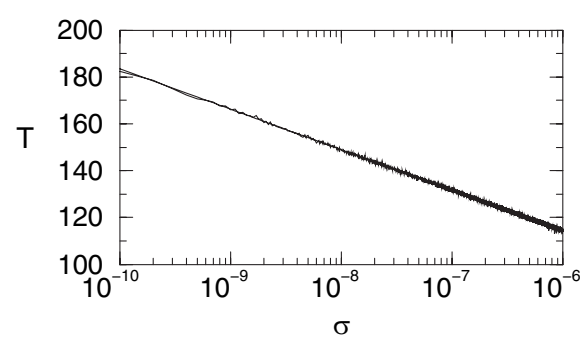

FIG. 3. Period dependence of the oscillations as a function of the noise level σ. The fit, a logarithmic function, is almost perfect with $T \propto \log(\sigma)/0.133$.

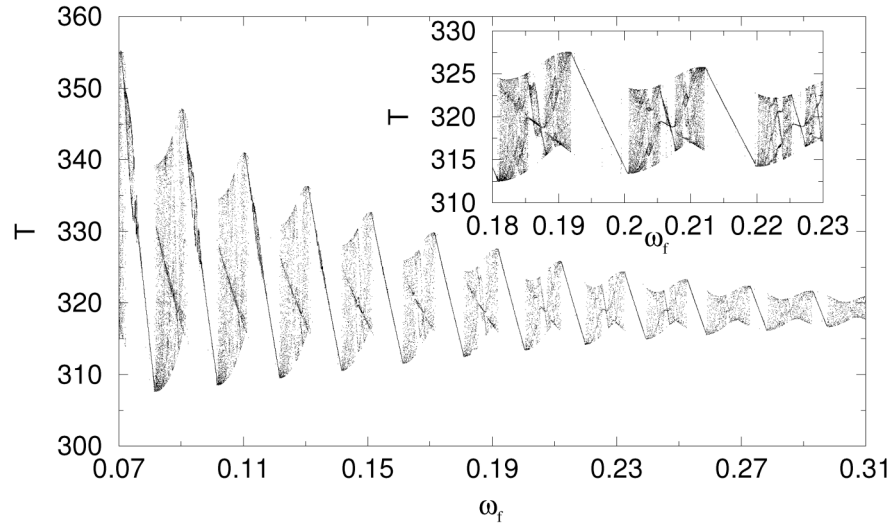

FIG. 4. Period as a function of the forcing frequency ω_f for $\gamma = 10^{-6}$ and $\sigma = 0$. The inset shows a blowup of the region $0.18 < \omega_f < 0.23$.

$$G(\Psi_n) = \begin{cases} g(\Psi_n) & \text{for } \Psi_n \leq D_3, \\ g(2\pi - \Psi_n) & \text{for } \Psi_n > D_3, \end{cases} \quad (4)$$

where $g(\Psi_n) = \pi(1 + 2j)/3 - b^{-1} + e^{b(\Psi_n - \pi(1+2j)/3 - b^{-1}\ln(b))}$, with $b = 4$ and j runs from 0 to 2 representing each of the exponential domains. For $j = 0$, $D_1 \geq \Psi_n > D_3$; for $j = 1$, $D_1 < \Psi_n \leq D_2$; and for $j = 2$, $D_2 < \Psi_n \leq D_3$. The domain connection points are $D_1 = \pi + b^{-1}\ln\{2/3[\pi b/(e^{2/3\pi b} - 1)]\}$, $D_2 = \pi + b^{-1}\ln\{2/3[\pi b/(e^{-2/3\pi b} - 1)]\}$, and $D_3 = 5\pi/3 + b^{-1}\ln(b) + b^{-1}\ln[(3 - 5\pi b)/3b]$. This map without forcing is shown in the inset of Fig. 5. The forcing function used to build the devil's staircase in this example is $F(\Psi_n) = 1 + \sin(\omega_f t)$ with $\beta = 0.1$.

Discussion.—The origin of the observed phenomenon is related to the following important features of the heteroclinic oscillator: (i) The oscillation frequency depends on the intensity of small perturbations as in (2) (see Fig. 3), and (ii) the oscillator is strongly nonlinear and the spectrum of the oscillations is very rich. These features are able to explain both the sensitive dependence of the ultrasubharmonic synchronization on the strength of a high periodic forcing, and the phase-locking itself.

This kind of phase synchronization can be observed in neurophysiological data. A popular point of view is that the gamma rhythm (40–80 Hz) plays the role of a timer in brain activity. The mechanisms of synchronization of low-frequency rhythms, i.e., theta (4–8 Hz) and beta (10–15 Hz), by gamma oscillation are not known yet. Since a possible mechanism for low-frequency rhythm generation in the brain is the activity of inhibitory competitive networks [23,24], it is reasonable to hypothesize that the phase synchronization of high- and low-frequency oscillations can be explained by the phenomenon discussed above. Phase synchronization among low- and high-frequency rhythms in the human cortex has been observed in Refs. [25,26]. Examples of frequency coordination of slow rhythm by fast rhythms have been observed also in specific networks of inhibitory neurons in invertebrates [27,28]. Robust synchronization of different neural rhythms can also play an important role in short-term memory [29,30]. For example, in Ref. [29] it has been shown that neural patterns associated with multiple short time memories can be stored in a single neural network that exhibits nested oscillations similar to those recorded from the brain. According to this multiplexing mechanism, each memory (or each event) is stored in a different high-frequency (~40 Hz) subcycle of a low-frequency (5–12 Hz) oscillation. In order to control the serial processing of short-term memories, these high- and low-frequency rhythms have to be synchronized.

The discussed phenomena can be of interest in many other disciplines, e.g., the study of the control of structured hydrodynamics by high-frequency acoustic forcings, the

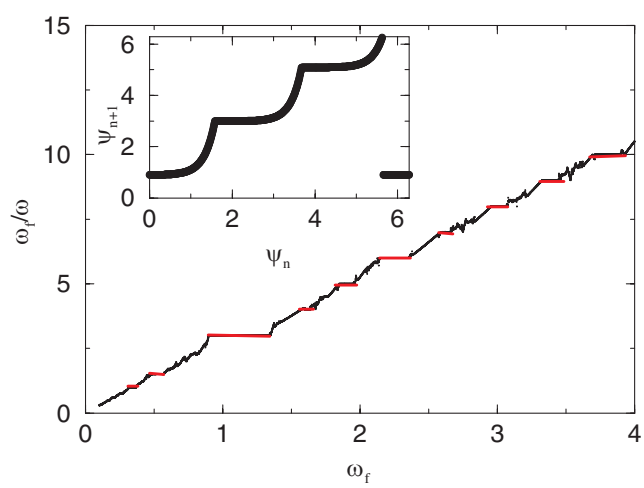

FIG. 5 (color online). Devil's staircase for the heteroclinic circle map (3), where ω is the frequency of oscillation of the system and ω_f is the forcing frequency. The inset figure represents the map (4) without forcing.

stabilization of chaotic dynamics by external forcings in lasers with heteroclinic cycles, or population dynamics of competitive species under seasonal perturbations.

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