Stability Criterion for Dissipative Soliton Solutions of the One-, Two-, and Three-Dimensional Complex Cubic-Quintic Ginzburg-Landau Equations

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The generation and nonlinear dynamics of multidimensional optical dissipative solitonic pulses are examined. The variational method is extended to complex dissipative systems, in order to obtain steady state solutions of the $(D+1)$-dimensional complex cubic-quintic Ginzburg-Landau equation ($D = 1, 2, 3$). A stability criterion is established fixing a domain of dissipative parameters for stable steady state solutions. Following numerical simulations, evolution of any input pulse from this domain leads to stable dissipative solitons.

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There is growing interest for optical solitons as form preserving self-confined structures. Temporal solitons may soon become the principal carrier in telecommunications in dispersion compensated optical fiber transmission systems [1,2]. Spatiotemporal solitons are good candidates in all-optical signal processing since they are self-guided in bulk media carrying big power for a small dissipated energy [3,4]. The stable operation of laser systems, closely related to the issue of dissipative soliton stability, is crucial for generating ultrashort pulses [5–7].

In order to generate a one-parameter or few-parameters family of solitons with transverse dimension $D = 1, 2, 3$, the diffraction and/or dispersion have to be compensated by spatial and/or temporal self-focusing [8]. However, real systems are generally dissipative; thus, linear and nonlinear gain and loss have to be taken into account. Dynamics of dissipative solitons can be described by a $(D+1)$-dimensional nonlinear complex cubic-quintic Ginzburg-Landau equation (CQGLE) [9]

$$i\frac{\partial E}{\partial z} + \Delta E + |E|^2 E + \nu |E|^4 E = \mathbb{Q}. \quad (1)$$

E is the normalized complex envelope of the optical field, and $\Delta E = r^{1-D}\partial/\partial r(r^{D-1}\partial E/\partial r)$ is the D-dimensional Laplacian describing beam diffraction and/or anomalous group velocity dispersion. In order to prevent the wave collapse the saturating nonlinearity is required. Therefore, cubic and quintic nonlinearities have to have opposite signs; i.e., parameter ν is negative. Dissipative terms are denoted by $\mathbb{Q}$,

$$\mathbb{Q} = i\delta E + i\varepsilon |E|^2 E + i\mu |E|^4 E + i\beta \Delta E. \quad (2)$$

Depending on the sign of the parameter δ, the first term is either linear gain or loss. The cubic and quintic gain-loss terms contain, respectively, parameters ε and μ. The last term accounts for the parabolic gain if $\beta > 0$. A prerequisite for generation of dissipative solitons is a simultaneous balance of not only diffraction and/or dispersion with self-focusing but also gain with loss, reducing for a given set of parameters a family of solutions to a fixed solution. Except for particular sets of parameters, there are no exact analytical solutions of CQGLE [10]. One has to resort to computer simulations in order to investigate the solutions of such an equation. General dynamical properties of Eq. (1) are rather complex making analytical approximation highly desirable. An analytical approach is essential also as a framework for a stability criterion still missing for dissipative solitons [11].

In this Letter we extend the variation approach established for the dissipative nonlinear Schrödinger equation in Ref. [12], to complex systems described by CQGLE. Based on this variational approach and the method of Lyapunov exponents, a general stability criterion for dissipative D-dimensional solitons is established. Input pulses generated in the proposed domain of dissipative parameters evolve towards stable dissipative solitons, as numerical simulations of CQGLE confirm.

The total Lagrangian $\mathbb{L} = \mathbb{L}_c + \mathbb{L}_Q$ of the system described by Eq. (1) contains besides a conservative part

$$\mathbb{L}_c = r^{D-1}\left\{\frac{i}{2}\left(E\frac{\partial E^*}{\partial z} - E^*\frac{\partial E}{\partial z}\right) + |\nabla E|^2 - \frac{|E|^4}{2} - \frac{\nu |E|^6}{3}\right\} \quad (3)$$

and also a dissipative part

$$\mathbb{L}_Q = i r^{D-1}\left\{\delta |E|^2 - \beta |\nabla E|^2 + \frac{\varepsilon |E|^4}{2} + \frac{\mu |E|^6}{3}\right\}. \quad (4)$$

Following Hamilton's principle $\delta(\iint(\mathbb{L}_c + \mathbb{L}_Q) dz dr) = 0$, the extremum function $E(z, r)$ renders the Lagrangian integral stationary under the condition that the Euler-Lagrange equation corresponding to Eq. (1)

$$\sum_{\xi}\frac{d}{d\xi}\left(\frac{\partial\mathbb{L}}{\partial E_{\xi}^{*\prime}}\right)-\frac{\partial\mathbb{L}}{\partial E^{*}}=0 \tag{5}$$

(where $\xi = z, r$) holds. The trial function of Gaussian shape

$$E = A(z)\exp\left[-\frac{r^2}{2R(z)^2}+iC(z)r^2+i\Phi(z)\right] \tag{6}$$

is expressed as a functional of amplitude A, pulse width R, wave front curvature C, and phase Φ. Following Kantorovitch, constant parameters of the Rayleigh-Ritz method are substituted here by functions of an independent variable $\eta(z) = A(z), R(z), C(z), \Phi(z)$ [12]. Optimization of each of these functions gives one of four Euler-Lagrange equations averaged over transverse coordinates

$$\int dr\left\{\sum_{\xi}d/d\xi(\partial\mathbb{L}_c/\partial E_{\xi}^{*\prime})-\partial\mathbb{L}_c/\partial E^{*}\right\}$$
$$=2\,\mathrm{Re}\int dr\left\{\partial\mathbb{L}_Q/\partial E^{*}-\sum_{\xi}d/d\xi(\partial\mathbb{L}_Q/\partial E_{\xi}^{*\prime})\right\}\partial E^{*}/\partial\eta. \tag{7}$$

where Re denotes the real part. The averaged conservative Lagrangian is denoted by $L_c = \int dr\mathbb{L}_c$

$$\frac{d}{dz}\left(\frac{\partial L_c}{\partial\eta'}\right)-\frac{\partial L_c}{\partial\eta}=2\,\mathrm{Re}\int r^{D-1}dr\mathbb{Q}\frac{\partial E^{*}}{\partial\eta}\equiv Q_{\eta}. \tag{8}$$

In the dissipative case the power $P = A^2R^D$ is no longer a constant [8], as can be seen after the variation with respect to the phase

$$d(A^2R^D)/dz = 2\{\delta+\varepsilon A^2/\sqrt{2}^D+\mu A^4/\sqrt{3}^D - D\beta(R^{-2}+4C^2R^2)/2\}A^2R^D. \tag{9}$$

The remaining three Euler-Lagrange equations correspond to the variations with respect to the amplitude

$$\frac{d\psi}{dz}+\frac{D}{2}\left(\frac{dC}{dz}+4C^2\right)R^2+\frac{D}{2R^2}-\frac{A^2}{\sqrt{2}^D}-\frac{\nu A^4}{\sqrt{3}^D}=0, \tag{10}$$

the width

$$\left(\frac{dC}{dz}+4C^2\right)R^2-\frac{1}{R^2}+\frac{A^2}{2\sqrt{2}^D}+\frac{2\nu A^4}{3\sqrt{3}^D}=-4\beta C, \tag{11}$$

and the curvature

$$d(A^2R^{D+2})/dz = \{2\delta+\varepsilon A^2/\sqrt{2}^D+2\mu A^4/3\sqrt{3}^{\,D}+\beta[(2-D)-(8+4D)C^2R^4]/R^2+8C\}A^2R^{D+2}. \tag{12}$$

Only Eq. (10) is identical to the corresponding equation in the conservative case, since the dissipative term Q_A is zero [8]. In order to have a stable pulse background, the linear dissipation term has to correspond to loss; i.e., the parameter δ must always be negative $\delta = -|\delta|$ [11]. It is renormalized as follows $\delta_0 = |\delta|\underline{R}^2$ where $\underline{R} = (8/3)^{1/2}(4/3)^{D/4}$. All remaining dissipative parameters are divided by $|\delta|$ and renormalized in order to be expressed in a unique form valid for different dimensions D: $\varepsilon_0 = (3/4)^{(1+D/2)}\varepsilon/|\delta|$, $\mu_0 = (3/4)^{(2+D/2)}\mu/|\delta|$, and $\beta_0 = (3/4)^{(1+D/2)}D\beta/4|\delta|$. All other quantities are also renormalized: $R/\underline{R}$, $z/\underline{R}^2$, $\underline{R}^2C$, and $A/\underline{A}$ where $\underline{A} = (3/4)^{1/2}(3/2)^{D/4}$. Therefore, within variational approximation, to the partial differential CQGLE corresponds to a set of four coupled first order differential equations (ODEs)

$$dA/dz = \{(1+D/4)\varepsilon_0A^2 + (1+D/3)\mu_0A^4-2\beta_0R^{-2}-1\}\delta_0A - 2DCA\equiv S, \tag{13}$$

$$dR/dz = \{2\beta_0D^{-1}(R^{-2}-4R^2C^2) - \varepsilon_0A^2/2-2\mu_0A^4/3\}\delta_0R+4RC\equiv F, \tag{14}$$

$$\frac{dC}{dz}=\frac{1}{R^4}-\frac{A^2}{R^2}-\frac{\nu A^4}{R^2}-4C^2-8\frac{\delta_0\beta_0}{DR^2}C\equiv G, \tag{15}$$

and

$$\frac{d\Phi}{dz}=4\beta_0\delta_0C-\frac{D}{R^2}+\frac{4+D}{2}A^2+\frac{3+D}{2}\nu A^4\equiv\Omega. \tag{16}$$

The steady state solutions can be obtained from Eqs. (13)–(15) for vanishing derivatives of amplitude, width, and curvature. These variables are expanded with respect to the small parameter $\delta_0 \ll 1$; $R = R_0 + \mathcal{O}(\delta_0^2)$ and $A = A_0 + \mathcal{O}(\delta_0^2)$ where coefficients of odd powers are zero, as well as $C = C_1\delta_0 + \mathcal{O}(\delta_0^3)$. For curvature even powers are vanishing. Only terms up to δ_0 are kept. The lowest order width $R = A^{-1}(1+\nu A^2)^{-1/2}$ and the propagation constant $\Omega = 0.5A^2[(4-D)+\nu(3-D)A^2]$ depend only on the amplitude as in the conservative case [8]. Variationally obtained families of conservative steady state solutions for $D = 1, 2$, and 3 lie on D curves in Fig. 1. The family of solutions reduces, in the dissipative case, to a fixed double solution for a given set of dissipative parameters. Indeed, the amplitude as a steady state solution of Eqs. (13)–(15) has two discrete values A^+ and A^-

$$A^{\pm}=\sqrt{\frac{(\beta_0-\varepsilon_0)\pm\sqrt{(\beta_0-\varepsilon_0)^2+4(\mu_0-\nu\beta_0)}}{2(\mu_0-\nu\beta_0)}} \tag{17}$$

denoted, respectively, by a triangle and a diamond on curves $D = 1$, 2, and 3 for parameters $\varepsilon_0 = 19$, $\mu_0 = -23.5$, $\beta_0 = 1.5$, $\delta_0 = 0.001$, and $\nu = -1$. The existence of either unique solution A^+ or double solution ($A^- > A^+$)

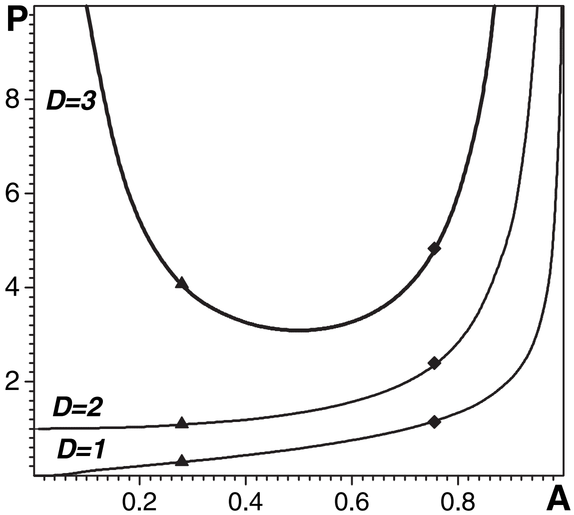

FIG. 1. Power P as a function of the amplitude A for one, two, and three dimensions.

implies a cubic gain $\varepsilon > 0$ and a quintic loss $\mu < 0$ (see Fig. 2). Unique solutions are separated in Fig. 2 from double solutions by the a line corresponding to $\mu_0 = 1 - \varepsilon_0 + \beta_0(1 + \nu)$. The domain of double solutions is also limited by the d parabola expressed as $(\beta_0 - \varepsilon_0)^2 + 4(\mu_0 - \nu\beta_0) = 0$. The double solution for the same sets of dissipative parameters as in Fig. 1 is again illustrated by a diamond superposed on a triangle. Another striking difference with conservative systems is the nonzero wave front curvature $C = \delta_0 A^2\{\varepsilon_0/8 - \beta_0/2D + (\mu_0/6 - \nu\beta_0/2D)A^2\}$ [8]. The gain-loss balance together with the compensation of diffraction and/or dispersion with saturating nonlinearity can be realized only for nonzero curvature fixed steady state solutions.

Only stable solutions can be solitons. Variationally obtained Euler-Lagrange equations are the starting point in order to establish a stability criterion using the method of Lyapunov's exponents [13]. A Jacoby determinant is constructed from derivatives with respect to amplitude, width, and curvature of terms S, F, and G of Eqs. (13)–(15) taken in steady, i.e., equilibrium, state

$$\det(J - \lambda I) = \left| \begin{matrix} \frac{\partial S}{\partial A} - \lambda & \frac{\partial S}{\partial R} & \frac{\partial S}{\partial C} \\ \frac{\partial F}{\partial A} & \frac{\partial F}{\partial R} - \lambda & \frac{\partial F}{\partial C} \\ \frac{\partial G}{\partial A} & \frac{\partial G}{\partial R} & \frac{\partial G}{\partial C} - \lambda \end{matrix} \right|_{\mathrm{eq}} = 0. \quad (18)$$

Following Lyapunov, steady state solutions of the set of nonlinear ODEs are stable if and only if the real part of solutions λ of cubic equation

$$\lambda^3 + \alpha_1\lambda^2 + \alpha_2\lambda + \alpha_3 = 0 \quad (19)$$

are negative [13]. In order to have Lyapunov's stability, Hurwitz's conditions must be fulfilled: the coefficients of Eq. (19), α_3 and α_2, as well as their combination $\alpha_1\alpha_2 - \alpha_3$, have to be positive. The nature of the stable steady state solution of nonlinear ODEs is determined by the condition $\alpha_5 = -\alpha_1^2\alpha_2^2 + 4\alpha_2^3 + 4\alpha_1^3\alpha_3 - 18\alpha_1\alpha_2\alpha_3 + 27\alpha_3^3 > 0$. The negative coefficient α_5 corresponds to the stable node while the positive one indicates the stable focus. The stability criterion for variationally obtained steady state solutions of D-dimensional CQGLE, up to δ_0, is explicitly expressed as follows:

$$\alpha_2 = 4A^4(1 + \nu A^2)[2 - D - 2\nu(D - 1)A^2] > 0, \quad (20)$$

$$\alpha_3 = 16A^4(1 + \nu A^2)[(\nu\varepsilon_0 - \mu_0)A^4 - 2\nu A^2 - 1]\delta_0 > 0, \quad (21)$$

and

$$\alpha_4 = \alpha_1\alpha_2 - \alpha_3 > 0, \quad (22)$$

where

$$\alpha_1 = \{-8/D + \varepsilon_0(8/D - 1 - D/2)A^2 + \mu_0(8/D - 8/3 - 4D/3)A^4\}\delta_0. \quad (23)$$

The coefficient α_3 is everywhere positive on the solution A^- and negative on A^+. As a consequence, only solutions A^- can be potentially stable. Above the tilted "horseshoe" b, c, and e corresponding to $\alpha_4 = 0$, the solution A^- of appropriate dimension is stable. This solution is a stable focus since $\alpha_5 = 4\alpha_2^3 > 0$. For instance, for the set of dissipative parameters from Figs. 1 and 2, the triangle on the lower unstable branch and the diamond on the upper stable branch of the v curve in Fig. 3 representing the amplitude as a function of a dissipative parameter ε_0, correspond, respectively, to A^+ and to A^-. Therefore, Eqs. (20)–(23) as stability criterion imply that any steady state solution of 1-, 2-, or 3-dimensional CQGLE belonging to the established stable domain of dissipative parameters will be stable. This criterion is tested using numerical

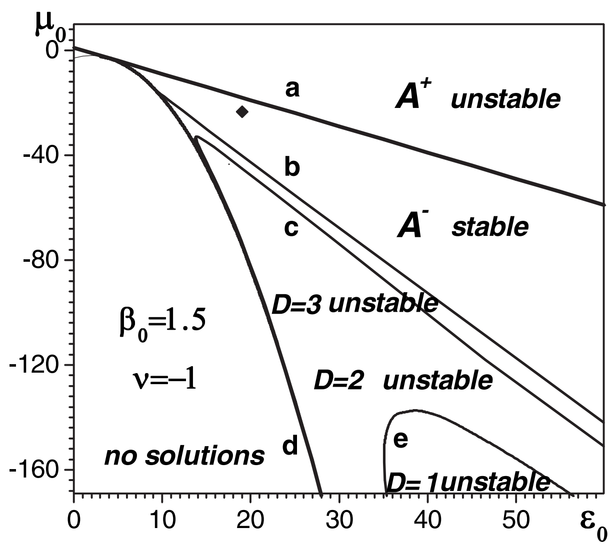

FIG. 2. Domain of stable solutions A^- for $D = 1$, 2, and 3.

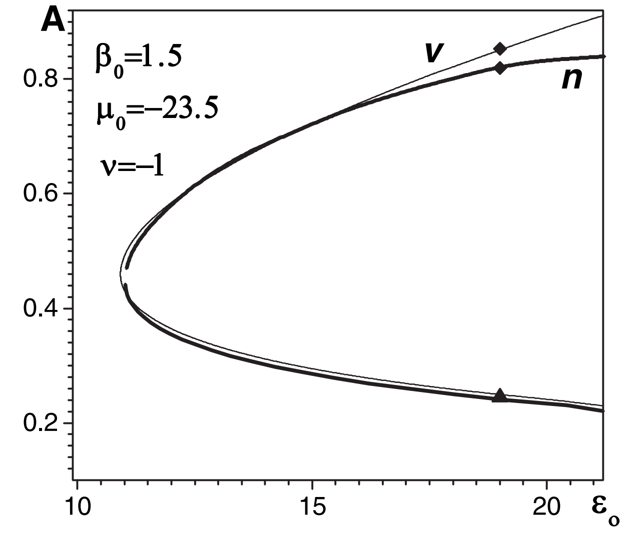

FIG. 3. Upper stable and lower unstable branches of variational v curve and numerical n curve.

simulations of CQGLE. Simulations of Eq. (1) are performed using the Crank-Nicholson integration scheme with the Gauss-Seidel iteration procedure. The integration step is $\Delta z = 0.01$. The number of sampling points is 201 following each transverse dimension. The input pulse chosen in the stable domain of parameters is not yet a stable soliton since the variationally obtained v curve does not coincide with the exact numerically obtained n curve in Fig. 3 (see two diamonds). Indeed, the variational approach gives only a good approximation. However, following our numerical simulations, the input pulse with parameters from the established stable domain evolves towards the stable dissipative soliton on the n curve. If the stable solution in Fig. 4(a) (corresponding to the diamond in Fig. 3) is taken as the input in numerical simulations, it will evolve towards the stable dissipative soliton in Fig. 4(b) tested till $z = 20\,000$. During evolution the amplitude slightly decreases in order to adjust to the exact soliton solution. Therefore, whenever an input pulse belongs to the stable domain, the final stage of evolution is always a stable dissipative soliton. Following Prigogine's theory of dissipative structures and self-organization, the curve in Fig. 3 can be interpreted as a bifurcation curve with an upper stable branch and a lower unstable branch with the control parameter ε_0 [13]. The generated dissipative structure, which is self-maintained against dissipation, is a stable dissipative soliton.

In conclusion, in order to obtain steady state solutions of the CQGLE, an analytical approach is developed based on the extension of the variational method to cubic-quintic dissipative systems. In order to treat simultaneously all three dimensions, the D-dimensional Laplacian in CQGLE has to be centrosymmetrical excluding asymmetric input pulses. However, for conservative systems, we

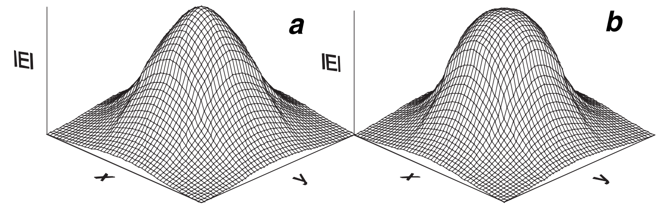

FIG. 4. Numerical evolution of an input pulse (a) towards a stable dissipative soliton (b).

demonstrated that asymmetric input pulses are driven by the cubic-quintic nonlinear Schrödinger equation toward the stable steady state solitonic solutions symmetric with respect to the transverse coordinates, the same as those obtained using *ab initio* a symmetric D-dimensional Laplacian [14]. An extension to the asymmetric CQGLE is the subject of a forthcoming paper. Such an analytical framework allowed us to establish for the first time a stability criterion based on the method of Lyapunov's exponents. The choice of input pulses with dissipative parameters belonging to the stability domain fixed by this criterion ensures the generation of stable dissipative solitons. This stability criterion opens the way to different practical applications in the conception of all-optical transmission systems, signal processing, and mode-locked laser generating ultrashort pulses.

[1] I. Gabitov and S. K. Turitsyn, Opt. Lett. **21**, 327 (1996).
[2] M. Nakazawa and H. Kubota, Jpn. J. Appl. Phys. **34**, L681 (1995).
[3] R. McLeod, K. Wagner, and S. Blair, Phys. Rev. A **52**, 3254 (1995).
[4] L.-C. Crasovan, B. A. Malomed, and D. Mihalache, Phys. Rev. E **63**, 016605 (2001).
[5] H. A. Haus, J. G. Fujimoto, and E. P. Ippen, J. Opt. Soc. Am. B **8**, 2068 (1991).
[6] C.-J. Chen, P. K. A. Wai, and C. R. Menyuk, Opt. Lett. **19**, 198 (1994).
[7] C. De Angelis, M. Santagiustina, and S. Wabnitz, Opt. Commun. **122**, 23 (1995).
[8] V. Skarka, V. I. Berezhiani, and R. Miklaszewski, Phys. Rev. E **56**, 1080 (1997).
[9] N. N. Akhmediev and A. Ankiewicz, *Solitons, Nonlinear Pulses and Beams* (Chapman and Hall, London, 1997).
[10] N. N. Akhmediev, V. V. Afanasjev, and J. M. Soto-Crespo, Phys. Rev. E **53**, 1190 (1996).
[11] J. M. Soto-Crespo, N. N. Akhmediev, and G. Town, Opt. Commun. **199**, 283 (2001).
[12] S. Chavez Cerda, S. B. Cavalcanti, and J. M. Hickmann, Eur. Phys. J. D **1**, 313 (1998).
[13] G. Nicolis and I. Prigogine, *Self-Organization in Nonequilibrium Systems* (John Wiley and Sons, New York, 1977).
[14] V. Skarka, V. I. Berezhiani, and R. Miklaszewski, Phys. Rev. E **59**, 1270 (1999).