Quantum Metrology

Vittorio Giovannetti,[1] Seth Lloyd,[2] and Lorenzo Maccone[3]

[1]*NEST-CNR-INFM & Scuola Normale Superiore, Piazza dei Cavalieri 7, I-56126, Pisa, Italy*

[2]*MIT, Research Laboratory of Electronics and Department of Mechanical Engineering, 77 Massachusetts Avenue, Cambridge, Massachusetts 02139, USA*

[3]*QUIT–Quantum Information Theory Group, Dipartimento di Fisica "A. Volta" Università di Pavia, via Bassi 6, I-27100 Pavia, Italy*

(Received 26 September 2005; published 3 January 2006)

We point out a general framework that encompasses most cases in which quantum effects enable an increase in precision when estimating a parameter (quantum metrology). The typical quantum precision enhancement is of the order of the square root of the number of times the system is sampled. We prove that this is optimal, and we point out the different strategies (classical and quantum) that permit one to attain this bound.

DOI: 10.1103/PhysRevLett.96.010401 PACS numbers: 03.65.Ta, 03.67.Lx, 06.20.Dk, 42.50.St

When estimating an unknown parameter in a quantum system, we typically prepare a probe, let it interact with the system, and then measure the probe. If the physical mechanism that governs the system dynamics is known, we can deduce the value of the parameter by comparing the input and the output states of the probe. Since quantum states are rarely distinguishable with certainty, there usually is an inherent statistical uncertainty in such estimation. To reduce this uncertainty, we can use N identical, independent probes, measure them, and average the results. From the central limit theorem, for large N the error on the average decreases as $\Delta/\sqrt{N}$, where Δ^2 is the variance of the measurement results associated with each probe. Using the same physical resources with the addition of quantum effects (such as entanglement or squeezing), an even better precision can often be achieved with a customary $\sqrt{N}$ enhancement, i.e., a scaling of $1/N$ [1].

In this Letter we introduce a theoretical framework that encompasses all of these strategies, and we show that the scaling $1/N$ is the general lower bound to the estimation error: The only way to further decrease the error is to reduce Δ, by improving the probe response to the interaction with the system. In analogy to quantum communication [2], different scenarios are possible (see Fig. 1): Either we do not employ quantum effects [classical-classical (CC) strategy] or quantum effects can be used only in the probe measurement [classical-quantum (CQ) strategy] or only in the probe preparation [quantum-classical (QC) strategy] or in both stages [quantum-quantum (QQ) strategy]. We show that the ultimate precision limit for the CC and CQ strategies is the classical limit $1/\sqrt{N}$, while the ultimate limit for the QC and QQ strategies is $1/N$. This means that, even though entanglement at the preparation stage is useful to increase the precision, it is useless at the measurement stage. Hence, the previously proposed methods for quantum-enhanced parameter estimation can be modified relinquishing entangled measurements without performance loss. Moreover, if one is willing to exchange physical resources with running time, the same precision $1/N$ of the quantum strategy can be achieved also classically by sequentially applying the transformation N times on the same probe (multiround protocol; see Fig. 2) [3,4]. We prove optimality also in this case: No multiround

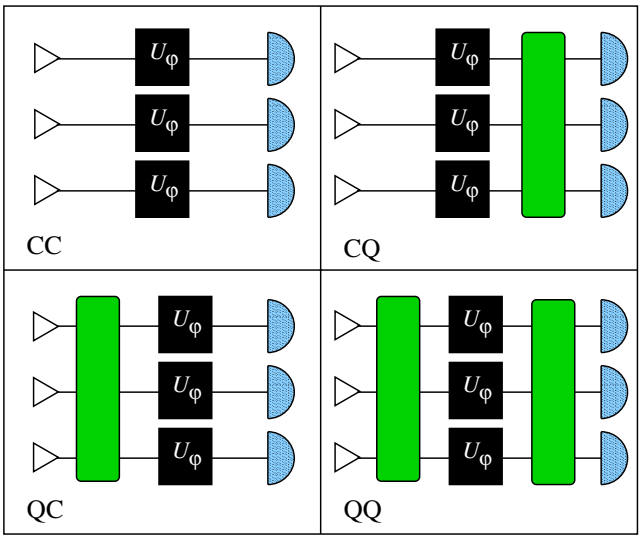

FIG. 1 (color online). Different strategies for the estimation of a parameter φ involving N parallel samplings of a unitary operator U_φ (black squares). The CC strategy involves separable input states and separable measurements [i.e., local operations and measurements whose results are communicated classically (LOCC)]. The CQ strategy involves separable input states and general measurement schemes. The QC strategy involves general input states (also entangled) and separable measurements. The QQ strategy involves general input states and general measurement schemes. The triangles on the left represent state preparation and the symbols on the right represent measurements. The gray boxes represent a unitary operation involving multiple probes (Q strategies).

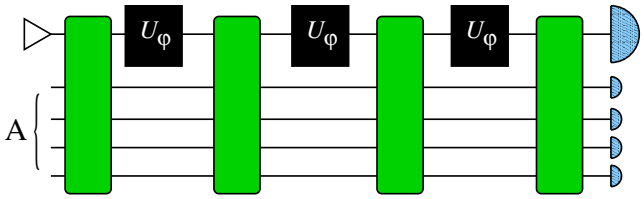

FIG. 2 (color online). Sequential (or multiround) protocol with a single probe. Thanks to the ancillary systems A, this scheme encompasses also adaptive techniques where information on φ is extracted between successive applications of the unitary U_φ. As in Fig. 1, the triangle represents the probe state preparation, the black squares represent U_φ, the symbols on the right represent detection, and the gray boxes represent unitary operations involving both the probe and the ancillas.

protocol exists that can achieve an error that scales better than $1/N$.

In the case of interferometry, it has long been claimed that, when using N photons in the interferometer, the Heisenberg limit $1/N$ is the ultimate bound to precision in phase measurements (whereas classical strategies permit one to reach only the shot noise limit of $1/\sqrt{N}$ [5]). However, the available proofs are based [6] on an incorrect interpretation of the time-energy uncertainty relation [7], or seem to lack the necessary generality as they refer [5,8] to specific interferometric setups. Our analysis clarifies that, indeed, the Heisenberg limit is the bound to interferometric precision. Our bound also applies to quantum phase-estimation strategies [9], which are customarily presented as examples of exponential-speedup algorithms. In fact, even though a precision $\sim 2^{-K}$ that scales exponentially with the number K of employed qubits is achieved, the algorithms require an exponential number of applications of the unitary U that generates the phase shift. Thus, in terms of the number $N \simeq 2^K$ of times that U needs to be employed in the procedure, one finds the same $1/N$ precision scaling of our optimality bound for sequential strategies.

In the following, we first analyze the theoretical framework that includes most known quantum metrology protocols. We then derive the bounds to precision in the different scenarios and show that they are achievable. Finally, we show how this relates to the known protocols and how to generate new protocols.

Theoretical framework.—Our goal is to find the most efficient possible way of estimating a parameter φ, introduced by the system onto the probe through a unitary operator $U_\varphi \equiv \exp(-i\varphi H)$, where the generator H is a known Hermitian operator. If we are allowed to sample the system N times, we can either use the parallel configuration of Fig. 1 where N probes are jointly employed or use the sequential configuration of Fig. 2 where a single probe is employed N times (or a combination of these two strategies). Notice that the latter configuration is in principle more powerful than the former. In fact, a sequential strategy can simulate any other configuration that employs the same number of U_φ's, if we add appropriate ancillas and if we allow the total running time to increase.

We start by analyzing the parallel strategies. Taking $|\Psi\rangle$ as the state of the N probes, it will be transformed into $U_\varphi^{\otimes N}|\Psi\rangle$, where $U_\varphi^{\otimes N}$ is the unitary transformation generated by $h = \sum_{j=1}^{N} H_j$ (H_j acting on the jth probe). In order to take into account the possibility that φ can be estimated through a general (biased or unbiased) estimator, it is convenient to use the error estimate [10,11]

$$\delta\varphi \equiv \left\langle \left(\varphi_{\text{est}} \Big/ \left| \frac{\partial \langle \varphi_{\text{est}} \rangle_{\text{av}}}{\partial \varphi} \right| - \varphi \right)^2 \right\rangle_{\text{av}}, \tag{1}$$

where φ_{est} is the estimator employed and where the brackets $\langle\,\rangle_{\text{av}}$ denote statistical averaging (the purpose of the derivative $\partial\langle\varphi_{\text{est}}\rangle_{\text{av}}/\partial\varphi$ is simply to express both φ_{est} and φ in the same "units"). Whatever is the measurement scheme employed, the error $\delta\varphi$ is bounded by the generalized uncertainty relation [11],

$$\delta\varphi \Delta h \geq 1/(2\sqrt{\nu}), \tag{2}$$

where $(\Delta h)^2 = \langle h^2\rangle - \langle h\rangle^2$ is the variance of h on the input state $|\Psi\rangle$ of the N probes, and ν is the number of times the estimation is repeated. Equation (2) derives from the Cramer-Rao bound and is asymptotically achievable in the limit of large ν. It implies that the minimum error $\delta\varphi$ is obtained when Δh is maximum. If $|\Psi\rangle$ is separable (CC and CQ strategies), $\Delta h = [\sum_j \Delta^2 H_j]^{1/2}$ where $\Delta^2 H_j$ is the variance of H_j on the state of the jth probe. Hence the maximum Δh is achieved by preparing each probe in a state having maximum spread for H_j, i.e., the equally weighted superpositions of the eigenvectors $|\lambda_M\rangle$ and $|\lambda_m\rangle$ of H_j corresponding, respectively, to the maximum and minimum eigenvalues λ_M and λ_m. These states have $\Delta H_j = (\lambda_M - \lambda_m)/2$, so that for any state we find $\Delta h = [\sum_j \Delta^2 H_j]^{1/2} \leq \sqrt{N}(\lambda_M - \lambda_m)/2$. This, through Eq. (2), gives an optimal CC and CQ error of

$$\delta\varphi \geq 1/[\sqrt{\nu N}(\lambda_M - \lambda_m)]. \tag{3}$$

This bound can be attained, for instance, by Ramsey interferometry, i.e., by preparing all the probes in the state $(|\lambda_M\rangle + |\lambda_m\rangle)/\sqrt{2}$, and by measuring the probability that each probe remains unchanged at the output. Even though Ramsey interferometry does not employ entangled measurements, these are accounted for in the derivation of Eq. (2); see [11]. This proves that entangled measurements are not necessary to achieve (3): The CC strategy is as accurate as the CQ strategy [12].

On the other hand, if $|\Psi\rangle$ can be entangled (QC and QQ strategies), the maximum Δh corresponds to a $|\Psi\rangle$, which is an equally weighted superposition of the eigenvectors relative to the maximum and minimum eigenvalues of the global generator h, i.e., $N\lambda_M$ and $N\lambda_m$. Since this state has

a spread $\Delta h_{\max} = (N\lambda_M - N\lambda_m)/2$, then a generic state has spread $\Delta h \leq N(\lambda_M - \lambda_m)/2$. This, through Eq. (2), gives an optimal QC and QQ error of

$$\delta\varphi \geq 1/[\sqrt{\nu}N(\lambda_M - \lambda_m)], \tag{4}$$

with a $\sqrt{N}$ improvement over Eq. (3). Notice that the derivation still applies if $|\Psi\rangle$ includes some external ancillas in addition to the probes, so that Eq. (4) accounts also for those detection strategies where half of an entangled state is fed into the system and a joint measurement is performed [13]. Also the bound (4) is attainable: Use the following entangled state of N probes:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\Big(|\lambda_m\rangle_1 \cdots |\lambda_m\rangle_N + |\lambda_M\rangle_1 \cdots |\lambda_M\rangle_N\Big), \tag{5}$$

and estimate φ by measuring the observable $X \equiv |\lambda_m\rangle\langle\lambda_M| + |\lambda_M\rangle\langle\lambda_m|$ separately on each probe at the output (an LOCC strategy). Since $\langle X^{\otimes N}\rangle_{\text{out}} = \cos[N\varphi(\lambda_M - \lambda_m)]$ and the variance $\Delta X^{\otimes N} = |\sin[N\varphi(\lambda_M - \lambda_m)]|$, after repeating the experiment ν times the error on φ can be obtained easily from error propagation as

$$\delta\varphi = \frac{1}{\sqrt{\nu}}\Delta X^{\otimes N}\Big/\left|\frac{\partial\langle X^{\otimes N}\rangle}{\partial\varphi}\right| = \frac{1}{\sqrt{\nu}N(\lambda_M - \lambda_m)}. \tag{6}$$

This procedure attains the bound (4) and again employs only separable measurements. This proves that entangled measurements are not necessary to achieve (4): The QC strategy is as accurate as the QQ strategy. It has been pointed out that the above $\delta\varphi$ refers only to the determination of the last significant digits of φ [14]. If one wants to determine all digits of φ, the procedure must be changed, but the $1/N$ scaling persists. For example, one can use a single probe ν times to recover the first decimal digit of $\varphi/2\pi$. Then, one can entangle 10 probes and determine the second decimal digit, still with ν repetitions. Iterating, the jth decimal digit will need 10^j entangled probes. Thus, the total number of probes (employed ν times) to recover l decimal digits is $\sum_{j=0}^{l} 10^j = (10^{l+1} - 1)/9$: Almost all the probes (roughly a fraction $9/10$ of the total) are employed to determine the last digit only.

Instead of a parallel strategy on N probes, one can employ a sequential strategy on a single probe. In this case the generator h in Eq. (2) must be modified: Instead of referring to the unitary $U_\varphi^{\otimes N}$ acting on N probes, it now refers to a unitary W_φ, which contains N applications of U_φ on a single probe, i.e., $W_\varphi = V_N U_\varphi V_{N-1} U_\varphi \cdots V_1 U_\varphi V_0$. Here the V_j's are arbitrary unitary operators acting on the probe and, eventually, on ancillary systems that can be used in adaptive strategies to extract information during the estimation process (i.e., the gray boxes of Fig. 2). In this case, the generator of W_φ is $h \equiv i(\partial W_\varphi/\partial\varphi)W_\varphi^\dagger \equiv \sum_{j=1}^{N} H_j'(\varphi)$, where $H_j'(\varphi) \equiv V_j U_\varphi \cdots V_1 U_\varphi V_0 H V_0^\dagger U_\varphi^\dagger V_1^\dagger \cdots U_\varphi^\dagger V_j^\dagger$ (H being the generator of U_φ). Since all the H_j' have the same spectrum as H, then the maximum eigenvalue of h is upper bounded by $N\lambda_M$, while the minimum eigenvalue of h is lower bounded by $N\lambda_m$. Hence, $\Delta h \leq N(\lambda_M - \lambda_m)/2$, and Eq. (2) in this case implies

$$\delta\varphi \geq 1/[\sqrt{\nu}N(\lambda_M - \lambda_m)]. \tag{7}$$

It is identical to the QC-QQ bound of Eq. (4), even though it refers to a different physical situation. This bound is again achievable through Ramsey interferometry, by preparing the single probe in the state $(|\lambda_M\rangle + |\lambda_m\rangle)/\sqrt{2}$, applying to it the transformation U_φ^N, and measuring the probability that it remains unchanged. Notice that the same analysis is valid also when the U_φ's are applied to more than one probe, i.e., for the strategies that are intermediate between the parallel and the sequential ones.

The QC and QQ protocols may seem less appealing than the multiround protocol since they require entanglement among the N probes to achieve the same sensitivity. However, their parallelizable structure entails that their running time may be N times smaller than the running time of the (necessarily sequential) multiround protocol. This is one of the instances frequently encountered in quantum computation where entanglement can convert spatial resources into temporal resources.

The above analysis illustrates how entanglement permits the full exploitation of the Hilbert space of N probes, granting access to ''high-resolution states'' such as the one given in Eq. (5). In repeating the process ν times, we can then achieve a precision that scales as $1/(N\sqrt{\nu})$ for large ν. This is a purely quantum effect. In fact, in a classical setting there is no advantage in grouping the measurements into ν groups of N: The error will invariably scale as $1/\sqrt{N\nu}$, i.e., as the inverse of the square root of the total number of measurements.

Notice that all the bounds we derived imply that there is no lower bound to the error $\delta\varphi$ if the system energy is unbounded, as in the case of the electromagnetic field (i.e., if $\lambda_M = \infty$). However, in all practical measurement schemes the energy involved is necessarily finite, which limits in practice the achievable resolution.

Quantum metrology protocols.—Most quantum metrology protocols can be analyzed under the theoretical framework outlined above. In particular, interferometric strategies can be accounted for by identifying N with the total number of passes of the employed photons through the interferometer, and the generator h with the electromagnetic field Hamiltonian. Here, the $1/N$ scaling of the optimal precision coincides with the Heisenberg limit, and it is well known that such a limit can be attained through entangled or squeezed light at the input ports of the interferometer (e.g., see [8]), or through multiround protocols [3]. The quantum-positioning and clock-synchronization protocol [15] is an example of interferometric strategy where a $1/N$ scaling in the precision of localization is obtained using frequency-entangled or number-squeezed

photons in a parallel configuration. The same results can be achieved also in a sequential configuration by bouncing back and forth a single photon [14].

Our framework encompasses many other estimation strategies. An example is the quantum-frequency-standards procedure [6,16], where the collective behavior of entangled atoms is used to enhance the precision of frequency measurements. It can be analyzed in our framework by identifying H_j with the two-level Hamiltonian of each probe atom. In this context, it is interesting to note that, in agreement with the equivalence between the QC and QQ strategies, one can achieve the upper bound (4) measuring separately the population of each atom [14], without resorting to the entangled measurement of the original proposals.

Using our framework, it is also possible to design new quantum metrology protocols. For example, by entangling N particles in momentum, we can design a strategy to obtain a better precision in the measurement of their average position from position measurements on the single particle (notice that in Ref. [15] the average position was deduced from time-of-arrival measurements and not from position measurements).

Even though we assumed that the operator H (i.e., the generator of the unitary U_φ) is known, the bounds we derived are valid also if H is unknown. However, in this case it is not granted that such bounds are achievable: All our ''achievability'' protocols require the knowledge of the eigenstates of H. Nonetheless, at least in the case of the reference-frames transmission (a procedure to employ N spins in transmitting a reference frame to a distant party, in which H is not known because it is the object to be estimated), a protocol achieving a scaling of $1/N$ has been recently proposed [17].

Conclusions.—State preparation is the primary factor in boosting the precision of the parameter estimation, while entangled measurements are never necessary. A $\sqrt{N}$ precision enhancement over what can be attained with a classical parallel strategy is typically obtained by using an input state that is entangled on a basis of eigenstates of H (the generator of the unitary U_φ) and by measuring a set of projectors on a basis dual to that. Schematic: (1) entangle N probes on the basis of eigenstates of H; (2) let the probes interact with the system; (3) measure on a dual basis. Result: a $\sqrt{N}$ precision enhancement. This is clearly related to the fact that entangled states can evolve faster than unentangled configurations employing the same resources [18]. Alternatively, a multiround protocol can achieve the same optimal precision at the expense of a larger running time.

V. G. was in part supported by the EC under Contract No. IST-SQUBIT2; S. L. acknowledges financial support by ARDA, DARPA, ARO, AFOSR, NSF, and CMI; and L. M. acknowledges financial support by MIUR through ''Cofinanziamento 2003'' and by EC through ATESIT (Contract No. IST-2000-29681).

[1] For a recent review, see V. Giovannetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004).
[2] C. H. Bennett and P. W. Shor, IEEE Trans. Inf. Theory **44**, 2724 (1998).
[3] A. Luis, Phys. Rev. A **65**, 025802 (2002).
[4] T. Rudolph and L. Grover, Phys. Rev. Lett. **91**, 217905 (2003).
[5] Z. Y. Ou, Phys. Rev. A **55**, 2598 (1997).
[6] J. J. Bollinger, W. M. Itano, D. J. Wineland, and D. J. Heinzen, Phys. Rev. A **54**, R4649 (1996).
[7] Y. Aharonov and D. Bohm, Phys. Rev. **122**, 1649 (1961); Y. Aharonov, S. Massar, and S. Popescu, Phys. Rev. A **66**, 052107 (2002).
[8] B. Yurke, S. L. McCall, and J. R. Klauder, Phys. Rev. A **33**, 4033 (1986).
[9] A. Y. Kitaev, quant-ph/9511026; R. Cleve, A. Ekert, C. Macchiavello, and M. Mosca, Proc. R. Soc. A **454**, 339 (1998); D. S. Abrams and S. Lloyd, Phys. Rev. Lett. **83**, 5162 (1999).
[10] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).
[11] S. L. Braunstein, C. M. Caves, and G. J. Milburn, Ann. Phys. (N.Y.) **247**, 135 (1996); S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. **72**, 3439 (1994).
[12] Equation (3) applies also when the probes are prepared into a separable mixture, $\sigma = \sum_k p_k \rho_k^{(1)} \otimes \cdots \otimes \rho_k^{(N)}$. The best experimental situation is when the information on the random variable k is accessible. In this case, the global state of ν experimental runs can be written as $\bigotimes_k [\rho_k^{(1)} \cdots \rho_k^{(N)}]^{\otimes n_k}$, where $n_k = p_k \nu$ is the number of times in which the kth component is employed. Starting from this tensor product form, we can employ the above derivation to obtain again the bound (3), which hence applies also to separable mixed states.
[13] G. M. D'Ariano and P. Lo Presti, Phys. Rev. Lett. **86**, 4195 (2001); G. M. D'Ariano, M. G. A. Paris, and P. Perinotti, Phys. Rev. A **65**, 062106 (2002).
[14] M. de Burgh and S. D. Bartlett, Phys. Rev. A **72**, 042301 (2005).
[15] V. Giovannetti, S. Lloyd, and L. Maccone, Nature (London) **412**, 417 (2001); V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. A **65**, 022309 (2002).
[16] S. F. Huelga, C. Macchiavello, T. Pellizzari, A. K. Ekert, M. B. Plenio, and J. I. Cirac, Phys. Rev. Lett. **79**, 3865 (1997).
[17] G. Chiribella, G. M. D'Ariano, P. Perinotti, and M. F. Sacchi, Phys. Rev. Lett. **93**, 180503 (2004).
[18] V. Giovannetti, S. Lloyd, and L. Maccone, Europhys. Lett. **62**, 615 (2003).