Double-Gap Alfvén Eigenmodes: Revisiting Eigenmode Interaction with the Alfvén Continuum

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A new type of global shear Alfvén eigenmode is found in tokamak plasmas where the mode localization is in the region intersecting the Alfvén continuum. The eigenmode is formed by the coupling of two solutions from two adjacent gaps (akin to potential wells) in the shear Alfvén continuum. For tokamak plasmas with reversed magnetic shear, it is shown that the toroidicity-induced solution tunnels through the continuum to match the ellipticity-induced Alfvén eigenmode so that the resulting solution is continuous at the point of resonance with the continuum. The existence of these double-gap Alfvén eigenmodes allows for potentially new ways of coupling edge fields to the plasma core in conditions where the core region is conventionally considered inaccessible. Implications include new approaches to heating and current drive in fusion plasmas as well as its possible use as a core diagnostic in burning plasmas.

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Introduction.—It is known that in toroidal plasmas, ideal magnetohydrodynamic (MHD) equations possess global solutions associated with gaps in the shear Alfvén continuum [1–4]. Each gap plays the role of an effective potential well for the global eigenmode solution that is produced by the coupling of poloidal harmonics due to either toroidal tokamak geometry or the noncircularity of its cross section. Normally, global solutions are localized within one gap and it is generally believed that if the mode is propagating from the gap to the continuum it is strongly damped due to the interaction with the absorbing layers at the resonances with the continuum [3–5]. This assumption is fundamental to the current understanding of global Alfvén eigenmode solutions in fusion plasmas.

In this Letter we show that this commonly held view is incorrect in certain plasma regimes with important implications for fusion scale experiments. A new type of global shear Alfvén eigenmode [called double-gap Alfvén eigenmode (DGAE)] is found when two gaps are separated by a continuum associated with only one poloidal $m \equiv m_s$ harmonic, where it has a singular point. This situation is particularly important for stellarator configurations where it is assumed that global mode solutions are suppressed due to coupling between modes with different toroidal mode numbers [6]. In tokamak plasmas it is also common to have radial continuum patterns that would conventionally be considered to prevent the establishment of global eigenmodes [3,4]. Damping on the continuum may affect the stability of shear Alfvén eigenmodes in a tokamak reactor, which, in turn, affects the confinement of energetic fusion products, α particles. Also the absorption by the continuum prevents externally excited low frequency Alfvén waves from being used for plasma heating and current drive as only the plasma periphery is affected [7,8]. By contrast, DGAEs can potentially couple the plasma edge to the core with implications for fast ion transport, heating, and current drive, and the use of external antennas as a diagnostic of the plasma core.

The DGAE can be understood by analogy to the Shrödinger equation with two adjacent potential wells [9]. The only difference is that in our case two solutions are separated by a potential barrier with an absorbing layer, where if any combination of the single well solutions has finite amplitude at that layer the mode is damped. Qualitatively a nondamped eigenmode solution can be constructed since we require that the continuum is described by the singularity of only one harmonic at a given radial location. If the singular harmonic has a node at the point of its singularity then an undamped global eigenmode can exist. This will be confirmed by detailed numerical analysis hereafter.

In a tokamak plasma one well-known example of the gap eigenmode is the toroidicity-induced Alfvén eigenmode due to coupling between m and $m + 1$ poloidal harmonics. This coupling produces the localized solution [toroidicity-induced Alfvén eigenmodes (TAE) [1]] at the $q = (2m + 1)/2n$ surface, where q is the safety factor and n is the toroidal mode number. Another kind of gap in the Alfvén continuum is due to the ellipticity-induced coupling, which results in the ellipticity-induced Alfvén eigenmodes (EAEs) [2] (see also noncircular Alfvén eigenmodes [10]). Within the ideal MHD framework the interaction with the continuum and thus its contribution to the damping rate can be calculated by making use of the perturbation technique, which requires that the damping rate is smaller than the mode frequency [4]. A kinetic treatment of this problem is possible by including finite Larmor radius effects in order to resolve the nonideal resonant layer in which the kinetic shear Alfvén wave absorbs energy propagating from the ideal region [5]. For weakly damped EAE/TAE solutions to exist, the mode radial structure should be in the evanescent (low amplitude) region where it resonates with the continuum in order for the damping rate to be weak.

The particular case considered in this Letter corresponds to a tokamak plasma with a reversed magnetic safety factor profile. The shear Alfvén continuum for this case is shown in Fig. 1 where TAE, EAE, and beta-induced AE (BAE) [11] gaps are indicated as functions of the radial variable, chosen as the square root of the normalized poloidal field flux $\rho \equiv (\psi/\psi_0)^{1/2}$, ψ_0 is the flux value at the plasma edge. The frequency on this figure is normalized to the Alfvén frequency, $\omega_A = \nu(0)/q_a R_0$, where $\nu_A(0)$ is the central Alfvén velocity, q_a is the edge safety factor, and R_0 is the major radius of the plasma cross section geometrical axis. This gap and the new solution are obtained and will be analyzed numerically using the ideal MHD code NOVA [1,12]. When the EAE and TAE gap solutions interact with each other a new solution, DGAE, is formed. We elaborate on the formalism in order to understand why a singularity is not developed in the radial mode structure. In this case the energy between EAE and TAE parts of the DGAE solution is exchanged via the sideband $m_s \pm 1, \pm 2$ poloidal harmonics. In some sense the situation is also analogous to holding the string at the node of its oscillations so that the energy does not dissipate at the zero amplitude point of the node.

Numerical solution.—For efficient interaction of solutions corresponding to different gaps, we choose plasma parameters in such a way that parts of EAE and TAE gaps are aligned radially. The chosen safety factor profile is shown in Fig. 2 along with the plasma beta and density. Other plasma parameters used in the simulations are: a major radius of the geometrical center of $R_0 = 3$ m, a minor radius of $a = 1$ m, a plasma central beta of $\beta_p(0) = 5\%$, last closed magnetic surface ellipticity of 1.8, and a triangularity of 0.3. The numerical method of the ideal MHD code NOVA is based on the poloidal harmonic representation for the poloidal dependence of the solution and third order polynomial finite elements in the radial direction [12]. DGAE poloidal harmonic radial structure of the quantity $\rho\xi_n$, where ξ_n is the plasma displacement component normal to the magnetic surface, is shown in Fig. 3. The eigenmode has the resonance with the continuum at $\rho = 0.67$. As one can see from Figs. 3(a) and 3(b), where the radial structures of three dominant (near resonance) harmonics are magnified, the solution is regular everywhere despite this resonance. Other solutions obtained numerically, such as TAEs, typically show jumps at the resonances. The convergence study was performed by gradually changing the number of radial grid points from 150 to 250 and showed that the solution is not sensitive to the grid size. To understand why there are no singularities at the continuum, i.e., where coefficients in front of the second derivative vanish and how this affects DGAE damping, we perform the analysis of the solution in the next section.

Analysis.—In this section we study the interaction of the DGAE with the continuum by making use of a flux function [4] that is continuous through the resonance. It was shown in Ref. [4] that the interaction with the continuum results in jumps of the perturbed amplitudes and it was proven that such interaction leads to the damping. In other words, jumps in the mode structure depend on how strong the interaction with the continuum is. As we can see in our case, there are no jumps in the real mode structure that suggest the interaction with the continuum is weak. Examine the continuation of the solution into the complex plane.

Ideal MHD equations can be reduced to a system of second order differential equations, which in the matrix form [12] serve as a basis for the numerical procedure of the NOVA code:

$$\frac{d}{d\rho}\left(A\frac{d}{d\rho}\xi\right) + B\frac{d}{d\rho}\xi + C\xi + \frac{d}{d\rho}(D\xi) = 0, \quad (1)$$

where $A(\rho)$, $B(\rho)$, $C(\rho)$, $D(\rho)$ are matrixes dependent on

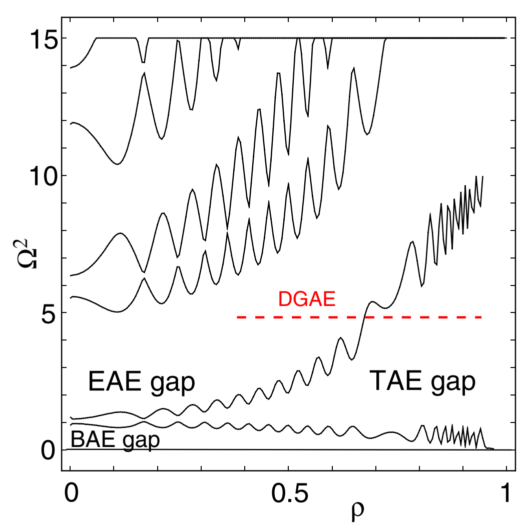

FIG. 1 (color online). Shear Alfvén continuum for $n = 8$ in the reversed shear plasma with the safety factor profile shown in Fig. 2. EAE, TAE, and BAE gaps are indicated. The DGAE solution radial extension is shown for the mode frequency $\Omega^2 = 4.8$.

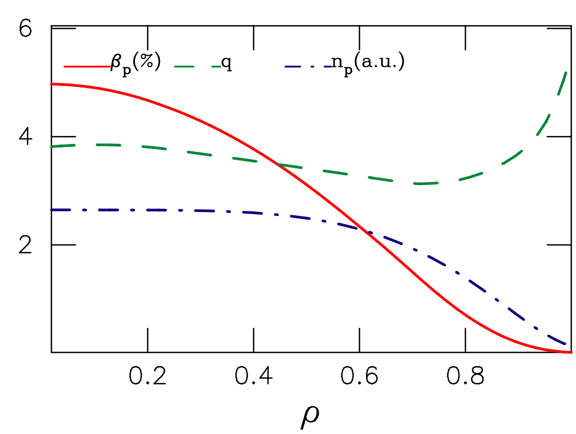

FIG. 2 (color online). The plasma safety factor, beta, and density profiles used in NOVA simulations.

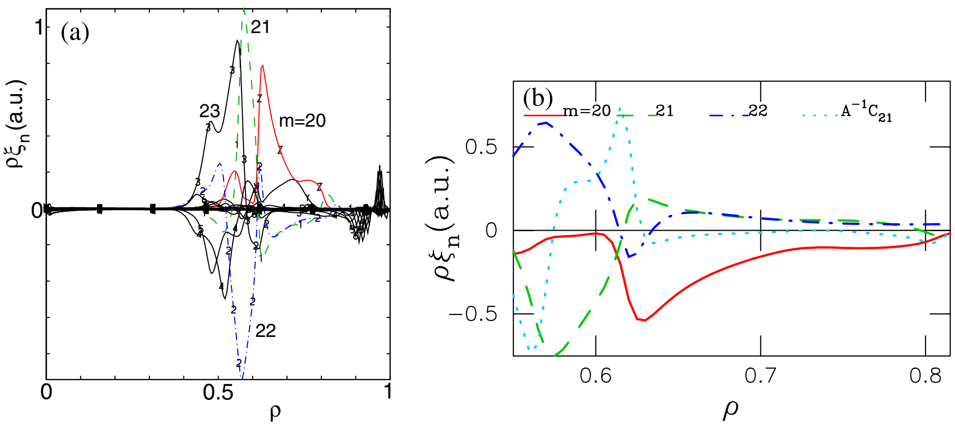

FIG. 3 (color online). DGAE poloidal harmonic radial structures for $n = 8$ (a). (b) Represents the magnified radial structure of $m = 20$–22 harmonics near the resonance with the continuum, $\rho = 0.67$. Also shown in (b) is $(A^{-1}C)_{m_s} = d\xi_m/d\rho$, $m_s = 21$.

the equilibrium plasma parameters and $\xi(\rho)$ is the vector of the amplitudes of the poloidal harmonics of the normal to the magnetic surface radial displacement of the plasma. Exact expressions of the matrixes from Eq. (1) are complicated [1,12] and are not required for the purpose of our analysis. One important property of matrix A is that the surface at which its determinant is vanishing ($\| A(\rho_s) \| = 0$) defines the location of the Alfvén continuum, $\rho_s = \rho_s(\Omega)$. We also note that $\| B + D \| = 0$ at ρ_s [1,12]. At this location the solution to Eq. (1) is also expected to be singular [5] and therefore to damp on the continuum. Figure 4 shows radial dependencies of three diagonal elements of matrix A, A_{mm}, $m = m_s$, $m_s \pm 1$, $m_s = 21$ in the vicinity of the resonant point. Calculations show that ρ_s in this case almost coincides with the solution of the equation $A_{m_s,m_s}(\rho) = 0$, which is $\rho = 0.67 \simeq \rho_s$.

Consider the introduction of a flux vector function similar to the one in Ref. [4], which, as we will see, is continuous across the resonance $\rho = \rho_s$:

$$\hat{C} \equiv A \frac{d}{d\rho} \xi. \tag{2}$$

This equation should be complemented by the equation for the flux following from Eq. (1):

$$\frac{d}{d\rho} \hat{C} = -B \frac{d}{d\rho} \xi - C\xi - \frac{d}{d\rho}(D\xi). \tag{3}$$

If we assume that $d \| A \| / d\rho \neq 0$ at ρ_s, then Eq. (2) can be formally inverted and by integrating it in the vicinity of ρ_s, one can conclude that ξ has a logarithmic singularity. From Eq. (3) it follows that $\hat{C}$ is indeed continuous at the resonance. This is confirmed in Fig. 4(b), which shows three elements, m_s, $m_s \pm 1$, of the flux vector having regular radial dependencies at the resonance. Note that the inversion of Eq. (2) gives a continuous radial derivative of ξ (despite vanishing $\| A \|$) shown in Fig. 3(b).

Nevertheless, regardless of the fact that the solution is continuous at the resonance, it is not clear whether the mode does not damp on the continuum. This is because the procedure of finding such damping requires an analytic continuation of the solution and its frequency onto the complex plane, whereas the code only finds the real solution and frequency.

Vector $\hat{C}$ can be used to show that in this case there is no continuum damping of the DGAE mode, i.e., the interaction with the continuum is weak. We invert Eq. (2) under the assumption of finite first derivative of the determinant of the matrix A at the resonance. The solution of Eq. (1) is found by the perturbative technique. Consider ξ_0 and ω_0

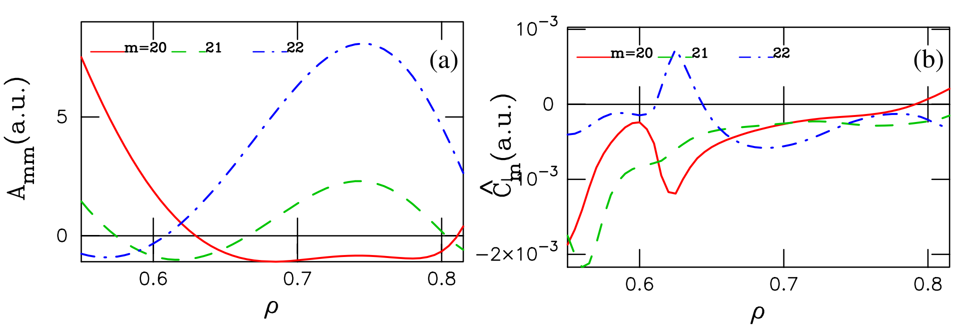

FIG. 4 (color online). Diagonal elements of the matrix of the coefficients at the second derivatives in the eigenmode equation, A_{mm}, $m = 20$–22, (a). (b) Represents flux vector-function elements, $\hat{C}_m$, for poloidal harmonics $m = 20, 21, 22$.

being the unperturbed solution and eigenfrequency and ξ_1 and ω_1 being their complex corrections, which take into account the jump in the solution at the continuum. Multiplying Eq. (1) by ξ from left, integrating it over the minor radius and requiring zero boundary condition, we obtain

$$\omega_1 = \left[\frac{\partial G(\omega_0)}{\partial \omega}\right]^{-1}[\xi_1(\rho_s + \varepsilon) - \xi_1(\rho_s - \varepsilon)]^T \hat{C}_0, \quad (4)$$

where $G(\omega_0) = \int d\rho[-(d\xi_0/d\rho)^T A d\xi_0/d\rho + (\xi_0)^T B d\xi_0/d\rho + (\xi_0)^T C\xi_0 - (d\xi_0/d\rho)^T D\xi_0]$, $\epsilon \ll \rho_s$ is small and is introduced to define the possible jump of the imaginary part of the solution at the resonance. To find it we integrate inverted Eq. (2) and make use of the causality condition, which for the imaginary part of the jump gives

$$\Im[\xi_1(\rho_s + \varepsilon) - \xi_1(\rho_s - \varepsilon)] = -\pi \frac{\|A\|_{\rho_s} A^{-1}\hat{C}_0}{|\partial\|A\|/\partial\rho|_{\rho_s}} \operatorname{sgn}(\partial\|A\|/\partial\omega)_{\rho_s},$$

so that Eq. (4) can be written in the form

$$\Im\omega_1 = -\pi\left[\frac{\partial G(\omega_0)}{\partial\omega}\right]^{-1} \times \frac{\|A\|_{\rho_s}\hat{C}_0^T A^{-1}\hat{C}_0}{|\partial\|A\|/\partial\rho|_{\rho_s}} \operatorname{sgn}(\partial\|A\|/\partial\omega)_{\rho_s}. \quad (5)$$

This allows us to compute the continuum damping. It follows from simulations that the damping is indeed small $\Im\omega_1/\omega < 10^{-5}$, which also means that the interaction with the continuum is weak.

Conclusions.—We have shown that the ideal MHD set of equations has solutions propagating across the continuum without interacting with it. The continuous solution at the resonance with the continuum results in vanishing continuum damping. This is opposite to the common understanding that the mode experiences strong interaction with the continuum if it has finite amplitude at the resonance. The condition for such modes to exist is that they should intersect the continuum only once in the vicinity of the resonance. Such resonance in the considered example is at the radial node of the eigenmode solution at the location of the singularity. This can be viewed as an eigenmode solution for the case of two adjacent potential wells separated by the barrier with localized absorbing layer. Note that the interaction with the continuum can be avoided even for the global TAE modes near the plasma edge if the TAE gap is closed in the radial direction.

We note that weakly damped DGAE cavity modes can reach into to the center of the plasma and hence can potentially be used for current drive and Alfvén wave heating for applications in laboratory and space plasmas. In addition, being global and weakly damped, DGAEs can be more unstable and easily excited by the super-Alfvénic fast ions, such as energetic fusion products, α particles, which, in turn, affects α-particle confinement. Interaction with the continuum also may be important for the stability of various Alfvén eigenmodes in stellarator plasma due to its complicated geometry and extra coupling between the modes with different toroidal mode numbers.

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