Imprint of Large-Scale Flows on Turbulence

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We investigate the locality of interactions in hydrodynamic turbulence using data from a direct numerical simulation on a grid of 1024^3 points; the flow is forced with the Taylor-Green vortex. An inertial range for the energy is obtained in which the flux is constant and the spectrum follows an approximate Kolmogorov law. Nonlinear triadic interactions are dominated by their nonlocal components, involving widely separated scales. The resulting nonlinear transfer itself is local at each scale but the step in the energy cascade is independent of that scale and directly related to the integral scale of the flow. Interactions with large scales represent 20% of the total energy flux. Possible explanations for the deviation from self-similar models, the link between these findings and intermittency, and their consequences for modeling of turbulent flows are briefly discussed.

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Flows in nature are often in a turbulent state driven by large-scale forcing (e.g., novae explosions in the interstellar medium) or by instabilities (e.g., convection in the sun). Such flows involve a huge number of coupled modes leading to great complexity both in their temporal dynamics and in the physical structures that emerge. Many scales are excited, for example, from the planetary scale to the kilometer for convective clouds in the atmosphere, and much smaller scales when considering microprocesses such as droplet formation. The large number of scales prohibits the use of direct numerical simulations to address these problems, and small scales tend to be modeled. The question then arises concerning the nature of the interactions between such scales: are they predominantly local, involving only eddies of similar size, or are they nonlocal as well? The answer to this question is crucial to construct trustworthy subgrid scale models of geophysical and engineering flows. It is usually assumed that the dominant mode of interaction is local, and this hypothesis is classically viewed as underlying the Kolmogorov phenomenology that leads to the prediction of a $E(k) \sim k^{-5/3}$ energy spectrum; such a spectrum has been observed in a variety of contexts although there may be small corrections to this power law due to the presence in the small scales of strong localized structures, such as vortex filaments [1].

Several studies have been devoted to assessing the degree of locality of nonlinear interactions, either through modeling of turbulent flows, as is the case with rapid distortion theory (RDT) [2] or large eddy simulations [3], or through the analysis of a direct numerical simulation (DNS) of the Navier-Stokes equations (see, e.g., [3–5]), and more recently through rigorous bounds [6]. The spatial resolution in the numerical investigations was moderate, without a clearly defined inertial range and the differentiation between local and nonlocal interactions was somewhat limited. Thus, a renewed analysis at substantially higher Reynolds numbers in the absence of any modeling is in order; we address this issue by analyzing data stemming from a newly performed DNS on a grid of 1024^3 points using periodic boundary conditions.

The governing Navier-Stokes equation for an incompressible velocity field $\mathbf{v}$, with $\mathcal{P}$ the pressure, $\mathbf{F}$ a forcing term, and $\nu = 3 \times 10^{-4}$ the viscosity, reads

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \mathcal{P} + \nu \nabla^2 \mathbf{v} + \mathbf{F} \tag{1}$$

together with $\nabla \cdot \mathbf{v} = 0$. Specifically, we consider the swirling flow resulting from the Taylor-Green vortex [7]:

$$\mathbf{F}_{\mathrm{TG}}(k_0) = 2F \begin{bmatrix} \sin(k_0 x)\cos(k_0 y)\cos(k_0 z) \\ -\cos(k_0 x)\sin(k_0 y)\cos(k_0 z) \\ 0 \end{bmatrix}, \tag{2}$$

with $k_0 = 2$. This forcing generates cells that have locally differential rotation and helicity, although its net helicity is zero. The resulting flow models the fluid between two counter-rotating cylinders [7] and has been widely used to study turbulence, including studies in the context of the generation of magnetic fields through dynamo instability [8]. The Reynolds number based on the integral scale $L \equiv 2\pi \int E(k)k^{-1}dk/E \approx 1.2$ (where E is the total energy) is $R_e \equiv UL/\nu \approx 4000$, where U is the rms velocity. The Reynolds number based on the Taylor scale $\lambda \equiv 2\pi(E/\int k^2 E(k)dk)^{1/2} \approx 0.24$, is $R_\lambda \approx 800$. The simulation was run for ten turnover times (L/U).

The code uses a dealiased pseudospectral method, with maximum wave number $k_{\max} = 341$ and $k_{\max}\eta = 1.15$, where $2\pi\eta = 2\pi(\nu^3/\epsilon)^{1/4}$ is the dissipation scale and ϵ is the energy injection rate: the flow is sufficiently resolved since $1/\eta$ is within the boundaries of the wave numbers handled explicitly in the computation.

Details of the flow dynamics will be reported elsewhere; suffice it to say that the flow reproduces classical features of isotropic turbulence [9]: the energy spectrum is well

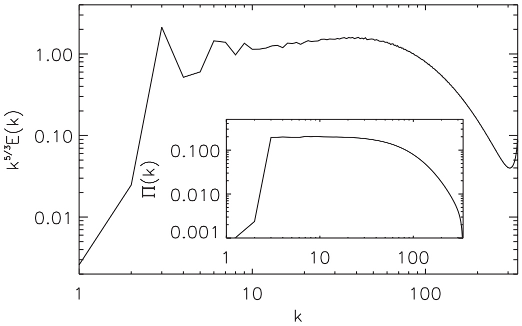

FIG. 1. Compensated energy spectrum and (inset) absolute value of the energy flux $\Pi(k)$ in the stationary regime.

developed (see Fig. 1) with a constant energy flux for $k \in [5, 20]$ and maximally helical vortex tubes are found, as predicted in Ref. [10] and shown in [11,12]. Finally, the anomalous exponents of longitudinal structure functions are within 10% agreement with previous studies [1] up to order $p = 8$ (see Table I), including analysis without using the extended self-similarity (ESS) hypothesis [13].

To investigate the interactions between different scales we split the velocity field into spherical shells in Fourier space of unit width, i.e., $\mathbf{v} = \sum_K \mathbf{v}_K$ where $\mathbf{v}_K$ is the filtered velocity field with $K \leq |k| < K + 1$ (from now on called shell K) [14]. From Eq. (1), the rate of energy transfer $T_3(K, P, Q)$ (a third-order correlator) from energy in shell Q to energy in shell K due to the interaction with the velocity field in shell P is defined as usual [15,16] as

$$T_3(K, P, Q) = -\int \mathbf{v}_K \cdot (\mathbf{v}_P \cdot \nabla)\mathbf{v}_Q d\mathbf{x}^3. \quad (3)$$

If we sum over the middle wave number P we obtain the total energy transfer $T_2(K, Q)$ from shell Q to shell K:

$$T_2(K, Q) = \sum_P T_3(K, P, Q) = -\int \mathbf{v}_K \cdot (\mathbf{v} \cdot \nabla)\mathbf{v}_Q d\mathbf{x}^3. \quad (4)$$

Positive transfer implies that energy is transferred from shell Q to K, and negative from K to Q; thus, both T_3 and T_2 are antisymmetric in their (K, Q) arguments (see Ref. [16]). $T_2(K, Q)$ gives information on the shell-to-shell energy transfer between K and Q, but not about the locality or nonlocality of the triadic interactions themselves. The energy flux plotted in Fig. 1 is reobtained from these transfer functions as $\Pi(k) = -\sum_{K=0}^{k} T_1(K) = -\sum_{K=0}^{k} \sum_Q T_2(K, Q)$. All transfer functions discussed were computed using one snapshot of the velocity field.

Figure 2 shows the energy transfer $T_2(K, Q)$ plotted as a function of $K - Q$ for 70 different values of Q varying from 10 to 80. For each value of Q, the x axis shows the different K shells giving or receiving energy from that shell Q. All curves collapse to a single one: the energy in shell K is received locally from shells with wave number $K - \Delta_K$ and deposited mostly in the vicinity of $K + \Delta_K$, with $\Delta_K \sim k_0$ for all values in the inertial range. This behavior has also been confirmed in smaller resolution runs (256^3) with different values of k_0. In other words, the integral scale of the flow, related to the forcing scale k_0^{-1}, plays a determinant role in the process of energy transfer. As a result, the transfer T_2 is not self-similar, and the integral length scale is remembered even deep inside the constant-flux inertial range.

This breakdown of self-similarity indicates that dominant triadic interactions can be nonlocal. To examine this point further, we need to investigate individual triadic interactions between Fourier shells by considering the tensorial transfer $T_3(K, P, Q)$. We will study three values of Q, ($Q = 10, 20$, and 40); for each case, P will run from 1 to 80, and K from $Q - 12$ to $Q + 12$.

In Fig. 3 we show contour levels of the transfer $T_3(K, P, Q)$ for $Q = 40$. This figure represents energy going from a shell Q to a shell K through interactions with modes in the shell P. As in Fig. 2, positive transfer means the shell K receives energy from the shell Q, while negative transfer implies the shell K gives energy to Q. The strongest interactions occur with $P \sim k_0$, and therefore the large-scale flow is involved in most of the T_2 transfer of energy from small scales to smaller scales. Note that the individual triadic interactions with $P \sim k_0$ and $K \sim Q \pm k_0$ are 2 orders of magnitude larger than local triadic interactions.

TABLE I. Order p and anomalous exponents ζ_p (defined by $\langle|\hat{\mathbf{r}} \cdot (\mathbf{u}(\mathbf{x} - \hat{\mathbf{r}}l) - \mathbf{u}(x))|^p\rangle = Cl^{\zeta_p}$, see Ref. [9]) averaged using two snapshots of the velocity field; the anomalous exponents ζ_p^{ESS} are computed from the same snapshots using the ESS hypothesis, and ζ_p^{SL} corresponds to the theoretical prediction by She and Lévêque [1].

p	1	2	3	4	5	6	7	8
ζ_p	0.37	0.70	1.00	1.27	1.50	1.70	1.88	2.03
ζ_p^{ESS}	0.364	0.694	1	1.270	1.505	1.695	1.881	2.031
ζ_p^{SL}	0.364	0.696	1	1.279	1.538	1.778	2.001	2.210

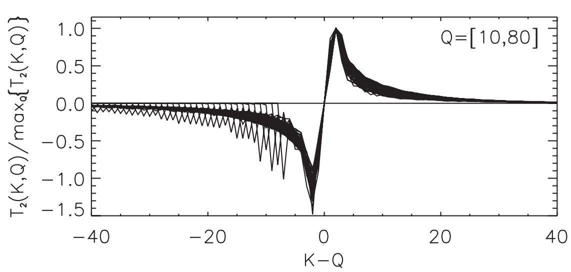

FIG. 2. Normalized energy transfer from the shell Q to the shell K. Each curve corresponds to a different value of Q in the range $Q \in [10, 80]$. The width of the lobes is independent of K and all the peaks are at $K - Q \sim \pm k_0$.

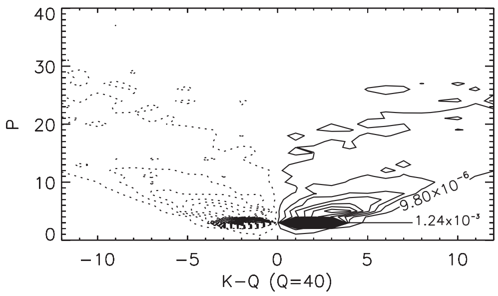

FIG. 3. Contour levels of the transfer function $T_3(K, P, Q)$ for $Q = 40$. Solid lines correspond to positive transfer, and dotted lines to negative transfer.

When $T_3(K, P, Q)$ in Fig. 4 is summed over all values of P, the transfer function $T_2(K, Q)$ is recovered. This allows us to define the transfer rate due to interactions with the large-scale flow, and due to local interactions, summing P over different ranges. Indeed, to further illustrate the dominance of the large-scale flow in the involved interactions, we compare in Fig. 4 the total transfer function $T_2(K, Q)$ with the transfer due to the large-scale flow $T_3(K, P = 3, Q)$, and with the transfer due to local interactions in octave bands $T_2^{\mathrm{Loc}}(K, Q) = \sum_{P=Q/2}^{2Q} T_3(K, P, Q)$. The figure indicates that the transfer due to the local interactions ($Q/2 < P < 2Q$) is smaller than the transfer due to the integral length scale velocity field, and this behavior appears to be stronger as the value of Q is increased. The remaining transfer comes from interactions with P shells with wave numbers between 1 and $Q/2$ (excluding $P = 3$), which are also nonlocal in nature. Therefore, as K and Q get larger (as we go further down in the inertial range), the dominant triads (K, P, Q) become more and more elongated, corresponding to more nonlocal interactions. As a result, detailed interactions between triads of modes are nonlocal, while the transfer of energy $T_2(K, Q)$ takes place between neighboring shells: local energy transfer occurs through nonlocal interactions. These results support previous claims at smaller resolution [3–5] that a significant role in the cascade of energy in the inertial range is played by the large-scale components of the velocity field.

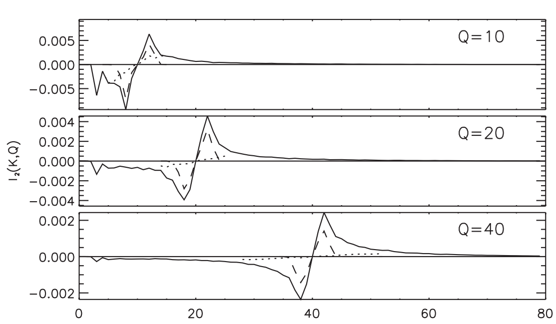

FIG. 4. Comparison of the transfer functions $T_2(K, Q)$ (solid line), $T_2^{\mathrm{Loc}}(K, Q)$ (dotted line), and $T_3(K, P = 3, Q)$ (dashed line), for three values of Q.

However, when computing the energy flux through a shell k, i.e., integrating $T_2(K, Q)$ over all values of Q, and K from 0 to k, these nonlocal interactions give ~20% of the total flux, since many more local triads contribute in the global summation. This fraction (20%) is independent of k, provided that k is large enough and in the inertial range.

We are left, therefore, with two puzzles. First, why is the large-scale flow more effective (at the level of individual triadic interactions) in "destroying" small size eddies than similar size eddies, when phenomenological arguments in the Kolmogorov spirit suggest otherwise? And secondly, why is the energy spectrum so close to $k^{-5/3}$ in the constant-flux region, when advection by the large-scale flow should give a shallower spectrum $\sim k^{-1}$ contribution? (see, e.g., Ref. [2]). In what follows, we give a brief review of possible answers as well as a simple model that shows how a $k^{-5/3}$ energy spectrum is also compatible with advection and stretching of the small scales just by the large-scale flow.

A possible answer to explain the strong nonlocal triadic interactions is that the Reynolds number in the present simulation is not high enough to observe dominance of local triads, and the decrease in amplitude of the small-scale fields due to viscosity makes these interactions (when compared to the large-scale flow) smaller. Another possible answer would be that the wave number bands defining the local interactions (i.e., the range of values in P used to define T_2^{Loc}), that were arbitrarily taken here to have a width of 2^n, could be as wide as 10^n as some authors suggest [5]. If this is the case, a DNS with an inertial range that spans at least 3 orders of magnitude in wave numbers would be required to actually observe strong local interactions.

However, neither of these answers addresses the second question concerning why a Kolmogorov energy spectrum is observed at moderate values of the Reynolds number. If we look at phenomenological scaling arguments, we see that there is one major assumption that may not be satisfied. Current models assume that the energy is distributed in a hierarchy of vortices of size $L, L/\alpha, L/\alpha^2, \ldots$ (with $\alpha > 1$), with no specific geometry. However, experiments as well as numerical simulations have shown that enstrophy is distributed in vortex tubes, where two distinct length scales can be identified: one is the width of the tube l that is typically small and varies, and one is its length L, typically of the order of the integral scale. It is not clear therefore when two such structures interact, which length scale is responsible for determining the time scale of the cascade.

From the analysis presented here, a simple model for turbulent flows consistent with several features observed in simulations and experiments can emerge (see below). First, recall that Ref. [12] found that helical vortex tubes capture 99% of the energy, give a $k^{-5/3}$ spectrum, and are responsible for the strong wings in the probability density function of velocity gradients. Furthermore, it was shown in Ref. [2] that, when decomposing the velocity field in a large-scale component U and a small-scale one u, artificially dropping local interactions in a simulation (an operation akin to RDT) gives enhanced intermittency (in the sense that a stronger departure from linear scaling of anomalous exponents is observed), while when nonlocal interactions are dropped the intermittency of the flow decreases [17].

The data analyzed in this Letter implies that, at low order of correlators, i.e., when considering the energy flux, the interactions are mostly local. But when going to third-order individual triadic interactions (such as with T_3), the nonlocal components are dominant and involve the integral scale. We note that this is consistent with the fact that departures from a linear scaling by anomalous exponents with the order of structure function is stronger as the order is increased, since it involves more nonlocal interactions linked to the geometrical structure of vortex tubes. This leads to a model of small-scale interactions involving three small scales that are substantially weakened and Gaussian, thus in agreement with the findings in Ref. [2] that such uu-like terms weaken intermittency as well when included in the full dynamics.

As a result, if we take into account the vortex tube structure of a turbulent flow, the picture of the classical Richardson cascade may change: a possible model to explain the aforementioned results is to take the time scale of the cascade as given by the geometric average of the length scales involved, based on the cubic root of the volume of the vortex tube. If this is the case, the energy dissipation rate of vortex tubes with velocity u_l due to the large-scale flow U_L is given by $\epsilon \sim u_l^2 U_L/(l^2 L)^{1/3}$. This implies that, for constant flux, $u_l \sim l^{1/3}\sqrt{\epsilon L^{1/3}/U_L}$, this scaling recovers the Kolmogorov spectrum, although in a different spirit [18]. Note that these arguments only show that both local interactions as well as nonlocal interactions can give a Kolmogorov scaling. The spirit of this derivation is close to multifractal models where the dimension of the structures in the flow are taken into account to explain intermittency corrections [9].

Finally, we would like to point out that the importance of the nonlocal interactions in a turbulent flow gives credibility to models involving as an essential agent of nonlinear transfer the distortion of turbulent eddies by a large-scale flow—as in RDT and its variants [2] or as in the alpha model [19] where the flow is interacting with a smooth velocity field (see also Ref. [20]). Similar results have already been obtained for flows coupled to a magnetic field, where nonlocality is expected to be stronger [21] and the second order transfer T_2 between velocity and magnetic field is nonlocal [16].

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[1] Z. S. She and E. Lévêque, Phys. Rev. Lett. **72**, 336 (1994); T. Kaneda *et al.*, Phys. Fluids **15**, L21 (2003); K. Yoshida, T. Ishihara, and Y. Kaneda, Phys. Fluids **15**, 2385 (2003).
[2] J.-P. Laval, B. Dubrulle, and S. Nazarenko, Phys. Fluids **13**, 1995 (2001); B. Dubrulle, J.-P. Laval, S. Nazarenko, and O. Zaboronski, J. Fluid Mech. **520**, 1 (2004).
[3] Y. Zhou, Phys. Fluids A **5**, 1092 (1993); **5**, 2511 (1993).
[4] J. A. Domaradzki and R. S. Rogallo, Phys. Fluids A **2**, 413 (1990); K. Ohkitani and S. Kida, Phys. Fluids A **4**, 794 (1992).
[5] J. G. Brasseur and C.-H. Wei, Phys. Fluids **6**, 842 (1994); P. K. Yeung, J. G. Brasseur, and Q. Wang, J. Fluid Mech. **283**, 43 (1995).
[6] G. L. Eyink, Physica (Amsterdam) **78D**, 222 (1994); G. L. Eyink, Physica (Amsterdam) **207D**, 91 (2005).
[7] M. Brachet, C. R. Acad. Sci. Paris Ser. IV **311**, 775 (1990); Fluid Dyn. Res. **8**, 1 (1991).
[8] Special issue on MHD dynamo experiments [Magnetohydrodynamics **38**, (2002)]; Y. Ponty *et al.*, Phys. Rev. Lett. **94**, 164502 (2005).
[9] U. Frisch, *Turbulence: The Legacy of A.N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).
[10] H. K. Moffatt, J. Fluid Mech. **150**, 359 (1985).
[11] A. Tsinober and E. Levich, Phys. Lett. A **99**, 321 (1983); E. Levich, Phys. Rep. **151**, 129 (1987).
[12] M. Farge, G. Pellegrino, and K. Schneider, Phys. Rev. Lett. **87**, 054501 (2001).
[13] R. Benzi *et al.*, Europhys. Lett. **24**, 275 (1993); R. Benzi *et al.*, Phys. Rev. E **48**, R29 (1993).
[14] With this definition, the force is acting on shell $K = 3$. Note also we are using a sharp filter in Fourier space, and as a result the bounds presented in Ref. [6] may not apply.
[15] R.H. Kraichnan, J. Fluid Mech. **47**, 525 (1971); M. Lesieur, *Turbulence in Fluids* (Kluwer Academic Press, Dordrecht, 1997).
[16] M. Verma, Phys. Rep. **401**, 229 (2004); A. Alexakis, P. D. Mininni, and A. Pouquet, Phys. Rev. E **72**, 046301 (2005); P. D. Mininni, A. Alexakis, and A. Pouquet, Phys. Rev. E **72**, 046302 (2005).
[17] Note that in agreement with these arguments, MHD, which is more nonlocal than neutral fluids [16,21] is also more intermittent as shown, e.g., in H. Politano, A. Pouquet, and V. Carbone, Europhys. Lett. **43**, 516 (1998).
[18] Note that this is consistent with a Beltramization of the flow quenching small-scale interactions. Using the Batchelor analogy between vorticity and induction, and the properties of the transfer in conducting fluids [16], we conjecture that the vorticity transfer itself will be nonlocal; this allows for the existence of a non-Beltrami component of nonlinear interactions which in turn could lead to the possible development of singularities.
[19] S. Y. Chen *et al.*, Phys. Fluids **11**, 2343 (1999); S. Y. Chen *et al.*, Physica (Amsterdam) **133D**, 66 (1999).
[20] D. Montgomery and A. Pouquet, Phys. Fluids **14**, 3365 (2002).
[21] P.S. Iroshnikov, Sov. Astron. **7**, 566 (1963); R.H. Kraichnan, Phys. Fluids **8**, 1385 (1965); A.P. Kazantsev, Sov. Phys. JETP **26**, 1031 (1968); R.M. Kulsrud and S.W. Anderson, Astrophys. J. **396**, 606 (1992); A.A. Schekochin *et al.*, Astrophys. J. **612**, 276 (2004).