Rare Decay $K^+ \to \pi^+ \nu\bar{\nu}$ at the Next-to-Next-to-Leading Order in QCD

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(Received 23 August 2005; published 27 December 2005)

We calculate the charm quark contribution to the rare decay $K^+ \to \pi^+ \nu\bar{\nu}$ in the next-to-next-to-leading order of QCD. This new contribution reduces the theoretical uncertainty in the relevant parameter P_c from $\pm 10.1\%$ down to $\pm 2.4\%$, corresponding to scale uncertainties of $\pm 1.3\%$, $\pm 1.0\%$, ± 0.006, and $\pm 1.2°$ in $\mathcal{B}(K^+ \to \pi^+ \nu\bar{\nu})$ and in $|V_{td}|$, $\sin 2\beta$, and γ extracted from the $K \to \pi\nu\bar{\nu}$ system. The error in $P_c = 0.37 \pm 0.04$ is now fully dominated by the current uncertainty of $\pm 3.8\%$ in the charm quark mass m_c. We find $\mathcal{B}(K^+ \to \pi^+ \nu\bar{\nu}) = (8.0 \pm 1.1) \times 10^{-11}$, where the quoted error stems almost entirely from the present uncertainties in m_c and the Cabibbo-Kobayashi-Maskawa elements.

DOI: 10.1103/PhysRevLett.95.261805 PACS numbers: 13.20.Eb, 12.15.Hh, 12.38.Bx

The rare process $K^+ \to \pi^+ \nu\bar{\nu}$ belongs to the theoretically cleanest decays in the field of K and B mesons. As it offers in conjunction with $K_L \to \pi^0 \nu\bar{\nu}$ a very clean determination of the standard unitarity triangle [1], a comparison of the information obtained from the $K \to \pi\nu\bar{\nu}$ system with the one from B decays provides a critical and truly unique test of the Cabibbo-Kobayashi-Maskawa (CKM) mechanism in the standard model (SM) [2,3]. Even if these K- and B-physics predictions agree, $K^+ \to \pi^+ \nu\bar{\nu}$ will allow one to discriminate between different extensions of the SM [2,3], by probing effective scales of new physics operators of up to a several TeV or even higher [4].

In the SM the decay $K^+ \to \pi^+ \nu\bar{\nu}$ proceeds through Z-penguin and electroweak box diagrams, which are sensitive to short-distance dynamics. As the required hadronic matrix elements can be extracted, including isospin breaking corrections [5], from the accurately measured leading semileptonic decay $K^+ \to \pi^0 e^+ \nu$, and the remaining long-distance contributions turn out to be small [6], and in principle calculable by means of lattice QCD [7], theoretical computations of the relevant decay rate can reach an exceptionally high degree of precision.

After summation over the three neutrino flavors the resulting branching ratio for $K^+ \to \pi^+ \nu\bar{\nu}$ can be written as [3,6,8,9]

$$\mathcal{B}(K^+ \to \pi^+ \nu\bar{\nu}) = \kappa_+ \left[\left(\frac{\mathrm{Im}\lambda_t}{\lambda^5} X(x_t) \right)^2 + \left(\frac{\mathrm{Re}\lambda_t}{\lambda^5} X(x_t) + \frac{\mathrm{Re}\lambda_c}{\lambda} (P_c + \delta P_{c,u}) \right)^2 \right], \tag{1}$$

$$\kappa_+ \equiv r_{K^+} \frac{3\alpha^2 \mathcal{B}(K^+ \to \pi^0 e^+ \nu)}{2\pi^2 \sin^4\theta_W} \lambda^8 = (5.04 \pm 0.17) \times 10^{-11} \left(\frac{\lambda}{0.2248} \right)^8. \tag{2}$$

Here $\lambda_i \equiv V_{is}^* V_{id}$ denote the relevant CKM factors, while $\delta P_{c,u} = 0.04 \pm 0.02$ encodes the long-distance contributions calculated recently in [6], and the parameter $r_{K^+} = 0.901 \pm 0.027$ summarizes isospin breaking corrections in relating $K^+ \to \pi^+ \nu\bar{\nu}$ to $K^+ \to \pi^0 e^+ \nu$ [5]. The apparent strong dependence of $\mathcal{B}(K^+ \to \pi^+ \nu\bar{\nu})$ on $\lambda \equiv |V_{us}|$ is spurious as P_c and $\delta P_{c,u}$ are proportional to $1/\lambda^4$. In quoting the value for P_c and $\mathcal{B}(K^+ \to \pi^+ \nu\bar{\nu})$ we will set $\lambda = 0.2248$ [10]. The electromagnetic coupling α and the weak mixing angle $\sin^2\theta_W$ entering $\mathcal{B}(K^+ \to \pi^+ \nu\bar{\nu})$ are naturally evaluated at the electroweak scale [11]. Then the leading term in the heavy top expansion of the electroweak two-loop corrections to $X(x_t)$ amounts to typically -1% for the modified minimal subtraction scheme ($\overline{\mathrm{MS}}$) definition of α and $\sin^2\theta_W$ [12]. In obtaining the numerical value of Eq. (2) we have employed $\alpha \equiv \alpha_{\overline{\mathrm{MS}}}(M_Z) = 1/127.9$, $\sin^2\theta_W \equiv \sin^2\hat{\theta}_W^{\overline{\mathrm{MS}}} = 0.231$, and $\mathcal{B}(K^+ \to \pi^0 e^+ \nu) = (4.93 \pm 0.07) \times 10^{-2}$ [13].

The function $X(x_t)$ in Eq. (1) depends on the top quark $\overline{\mathrm{MS}}$ mass through $x_t \equiv m_t^2(\mu_t)/M_W^2$. It originates from Z-penguin and electroweak box diagrams with an internal top quark. As the relevant operator has a vanishing anomalous dimension and the energy scales involved are of the order of the electroweak scale or higher, the function $X(x_t)$ can be calculated within ordinary perturbation theory. It is known through the next-to-leading order (NLO) [9,14], with a scale uncertainty due to the top quark matching scale $\mu_t = \mathcal{O}(m_t)$ of only $\pm 1\%$. Converting the top quark pole mass of $M_t = (172.7 \pm 2.9)$ GeV [15] at three loops to $m_t(M_t)$ [16] and relating $m_t(M_t)$ to $m_t(m_t) = (163.0 \pm 2.8)$ GeV using the one-loop renormalization group (RG), we find $X(x_t) = 1.464 \pm 0.041$. The given uncertainty combines linearly an error of ± 0.028 due to the error of $m_t(m_t)$ and an error of ± 0.013 obtained by varying μ_t in the range 60 GeV $\leq \mu_t \leq$ 240 GeV.

The calculable parameter P_c entering Eq. (1) results from Z-penguin and electroweak box diagrams involving internal charm quark exchange. As now both high- and low-energy scales, namely, $\mu_W = \mathcal{O}(M_W)$ and $\mu_c = \mathcal{O}(m_c)$, are involved, a complete RG analysis of this term is required. In this manner, large logarithms $\ln(\mu_W^2/\mu_c^2)$ are resummed to all orders in α_s. At the leading order such an analysis has been performed in [17]. The large scale uncertainty due to μ_c of $\pm 26\%$ in this result was a strong motivation for the NLO analysis of this contribution [8,9].

Performing the RG running from μ_W down to $\mu_b = \mathcal{O}(m_b)$ in an effective five-flavor theory and the subsequent evolution from μ_b down to μ_c in an effective four-flavor theory, we obtain at the NLO

$$P_c = 0.367 \pm 0.037_{\text{theor}} \pm 0.033_{m_c} \pm 0.009_{\alpha_s} = (0.37 \pm 0.06)\left(\frac{0.2248}{\lambda}\right)^4, \quad (3)$$

where the parametric errors correspond to the ranges of the charm quark $\overline{\text{MS}}$ mass $m_c(m_c)$ and the strong coupling constant $\alpha_s(M_Z)$ given in Table I. We note that the final error has only an illustrative character, since the partial uncertainties are not statistically distributed. Numerically, it is the mean of the value obtained by adding the individual errors once linearly and once in quadrature. The same way of combining errors will be applied in Eqs. (4), (8), and (9).

The dependence of P_c on μ_c can be seen in Fig. 1. The solid line in the upper plot shows the NLO result obtained by evaluating $\alpha_s(\mu_c)$ from $\alpha_s(M_Z)$ solving the RG equation of α_s numerically, while the dashed and dotted lines are obtained by first determining the scale parameter $\Lambda_{\overline{\text{MS}}}$ from $\alpha_s(M_Z)$, either using the explicit solution of the RG equation of α_s or by solving the RG equation of α_s iteratively for $\Lambda_{\overline{\text{MS}}}$, and subsequently calculating $\alpha_s(\mu_c)$ from $\Lambda_{\overline{\text{MS}}}$. The corresponding two-loop values for $\alpha_s(\mu_c)$ have been obtained with the program RUNDEC [19]. Obviously, the difference between the three curves is due to higher order terms and has to be regarded as part of the theoretical error. With its size of ± 0.012 it is comparable to the variation of the NLO result due to μ_c, amounting to ± 0.020. In [3,8,9] larger values for the latter uncertainty have been quoted. The observed difference is related to the definition of the charm quark mass. Replacing $m_c(m_c)$ in the logarithms $\ln(\mu_c^2/m_c^2)$ of the one-loop matrix elements by the more appropriate $m_c(\mu_c)$ leads to a significant reduction of the dependence of P_c on μ_c. A detailed discussion of this issue will be presented in [20]. Finally, while in [3,8,9] only μ_c was varied, the theoretical error given in Eq. (3) includes also the dependence on μ_b and μ_W of ± 0.004 and ± 0.001, respectively. The specified scale uncertainties correspond to the ranges $1\ \text{GeV} \le \mu_c \le 3\ \text{GeV}$, $2.5\ \text{GeV} \le \mu_b \le 10\ \text{GeV}$, and $40\ \text{GeV} \le \mu_W \le 160\ \text{GeV}$.

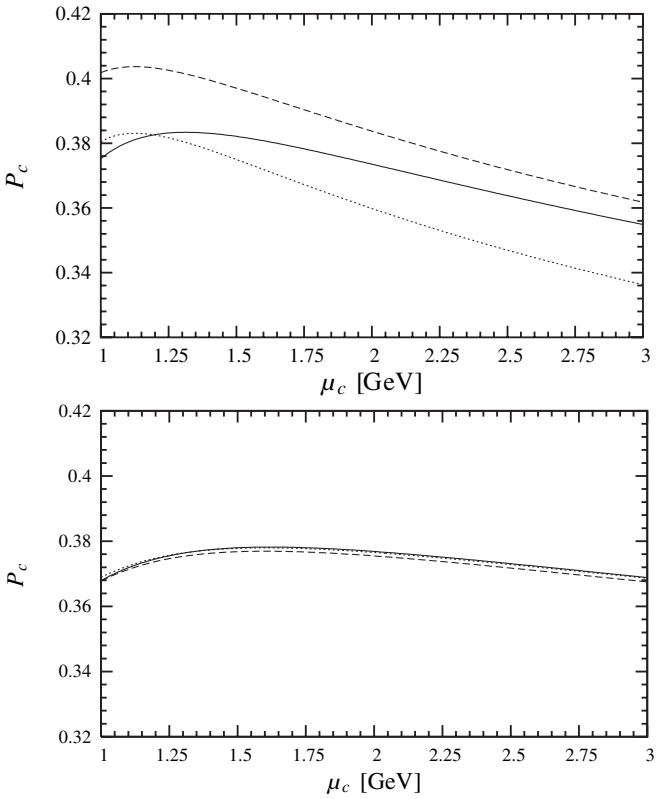

FIG. 1. P_c as a function of μ_c at NLO (upper plot) and NNLO (lower plot). The three different lines correspond to three different methods of computing $\alpha_s(\mu_c)$ from $\alpha_s(M_Z)$ (see text).

TABLE I. Input parameters used in the numerical analysis of P_c, $\mathcal{B}(K^+ \to \pi^+ \nu\bar{\nu})$, $|V_{\text{td}}|$, $\sin 2\beta$ and γ.

Parameter	Value $\pm$ Error	Reference
$m_c(m_c)$ [GeV]	1.30 ± 0.05	[18] (our average)
$\alpha_s(M_Z)$	0.1187 ± 0.0020	[13]
$\text{Im}\lambda_t[10^{-4}]$	$1.407^{+0.096}_{-0.098}$	[10]
$\text{Re}\lambda_t[10^{-4}]$	$-3.13^{+0.20}_{-0.17}$	[10]
$\text{Re}\lambda_c$	$-0.220\,06^{+0.000\,93}_{-0.000\,91}$	[10]

Using the input parameters listed in Table I, we find from Eqs. (1)–(3) at the NLO [21]

$$\mathcal{B}(K^+ \to \pi^+ \nu\bar{\nu}) = (7.93 \pm 0.77_{P_c} \pm 0.84_{\text{other}}) \times 10^{-11} = (7.9 \pm 1.3) \times 10^{-11}, \quad (4)$$

where the second error in the first line collects the uncertainties due to κ_+, $\delta P_{c,u}$, $X(x_t)$, and the CKM elements. The final error has only an illustrative character, as the individual uncertainties have no statistical interpretation. Numerically, the enhancement of $\mathcal{B}(K^+ \to \pi^+ \nu\bar{\nu})$ coming from $\delta P_{c,u}$ [6] has been compensated by the suppression due to the decrease of M_t [15].

Provided P_c is known with a sufficient precision, a measurement of $K^+ \to \pi^+ \nu\bar{\nu}$, either alone or together with one of $K_L \to \pi^0 \nu\bar{\nu}$, allows for precise determinations

of the CKM parameters [1]. The comparison of this unitarity triangle with the one from B physics offers a stringent and unique test of the SM. In particular, for $\mathcal{B}(K^+ \to \pi^+ \nu\bar{\nu})$ and $\mathcal{B}(K_L \to \pi^0 \nu\bar{\nu})$ close to their SM predictions, one finds that a given uncertainty $\sigma(P_c)$ translates into

$$\frac{\sigma(|V_{td}|)}{|V_{td}|} = \pm 0.41 \frac{\sigma(P_c)}{P_c}, \tag{5}$$

$$\frac{\sigma(\sin 2\beta)}{\sin 2\beta} = \pm 0.34 \frac{\sigma(P_c)}{P_c}, \tag{6}$$

$$\frac{\sigma(\gamma)}{\gamma} = \pm 0.83 \frac{\sigma(P_c)}{P_c}, \tag{7}$$

with similar formulas given in [3]. Here V_{td} is the element of the CKM matrix and β and γ are the angles in the standard unitarity triangle. As the uncertainties in Eqs. (3) and (4) coming from the charm quark mass and the CKM parameters should be decreased in the coming years it is also desirable to reduce the theoretical uncertainty in P_c. To this end, we here extend the NLO analysis of P_c presented in [8,9] to the next-to-next-to-leading order (NNLO). This requires the computation of three-loop anomalous dimensions of certain operators and of certain two-loop contributions.

The main components of the NNLO calculation [20], which aims at resumming all $\mathcal{O}(\alpha_s^n \ln^{n-1}(\mu_W^2/\mu_c^2))$ logarithms in P_c, are (i) the $\mathcal{O}(\alpha_s^2)$ matching corrections to the relevant Wilson coefficients arising at μ_W, (ii) the $\mathcal{O}(\alpha_s^3)$ anomalous dimensions describing the mixing of the dimension-six and -eight operators, (iii) the $\mathcal{O}(\alpha_s^2)$ threshold corrections to the Wilson coefficients originating at μ_b, and (iv) the $\mathcal{O}(\alpha_s^2)$ matrix elements of some of the operators emerging at μ_c.

Conceptual new features in our NNLO calculation are (a) the appearance of the vector component of the effective neutral-current coupling describing the interaction of neutrinos and quarks mediated by Z-boson exchange, (b) the presence of anomalous triangle contributions which make it necessary to introduce a Chern-Simons operator in order to obtain the correct anomalous Ward identity involving the axial-vector coupling of the Z boson, and (c) the existence of nontrivial two-loop matching corrections to the Wilson coefficients of the current-current operators at the bottom quark threshold.

To determine the contributions of types (i), (iii), and (iv) one must calculate two-loop Green functions in the full SM and in effective theories with five or four flavors. Sample diagrams for steps (i) and (iv) are shown in the left and right columns of Fig. 2. The contributions (ii) are found by calculating three-loop Green functions with operator insertions. Sample diagrams with a double insertion of dimension-six operators are shown in the center column of Fig. 2. The corresponding three-loop amplitudes are

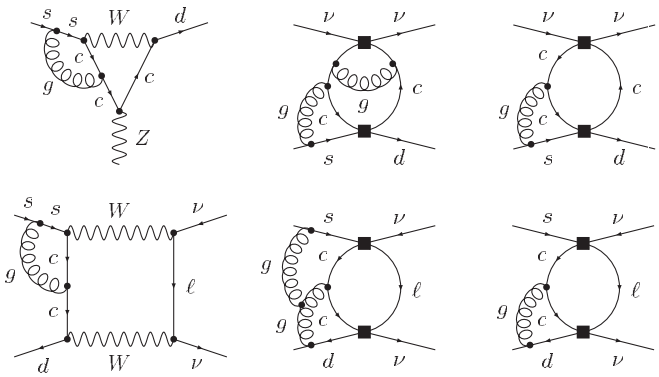

FIG. 2. Examples of Feynman diagrams arising in the full SM (left column), describing the mixing of operators (center column), and the matrix elements (right column) in the Z-penguin (upper row) and the electroweak box (lower row) sector. Only the divergent pieces of the diagrams displayed in the center column have to be computed, while the Feynman graphs shown on the left- and right-hand sides are needed, including their finite parts.

evaluated using the method that has been described in [22,23]. A comprehensive discussion of the technical details of the matching, the renormalization of the effective theory and the actual calculation will be given in [20].

Having described the general steps of our calculation, we now present our results. Using the general RG formalism [23,24], we find at the NNLO

$$\begin{aligned} P_c &= 0.371 \pm 0.009_{\text{theor}} \pm 0.031_{m_c} \pm 0.009_{\alpha_s} \\ &= (0.37 \pm 0.04)\left(\frac{0.2248}{\lambda}\right)^4, \end{aligned} \tag{8}$$

where the final error is fully dominated by the uncertainty in $m_c(m_c)$. Comparing these numbers with Eq. (3) we observe that our NNLO calculation reduces the theoretical uncertainty by a factor of 4.

As can be nicely seen in the lower plot of Fig. 1, P_c depends very weakly on μ_c at the NNLO, varying by only ± 0.0047. Furthermore, the three different treatments of α_s affect the NNLO result by as little as ± 0.0005. The three-loop values of $\alpha_s(\mu_c)$ used in the numerical analysis have been obtained with the program RUNDEC [19]. The theoretical error quoted in Eq. (8) includes also the dependence on μ_b and μ_W of ± 0.0028 and ± 0.0007, respectively. The presented scale uncertainties correspond to the ranges given earlier.

Using Eqs. (1), (2), and (8) the result in Eq. (4) is modified to the NNLO value [21]

$$\begin{aligned} \mathcal{B}(K^+ \to \pi^+ \nu\bar{\nu}) &= (7.96 \pm 0.49_{P_c} \pm 0.84_{\text{other}}) \times 10^{-11} \\ &= (8.0 \pm 1.1) \times 10^{-11}. \end{aligned} \tag{9}$$

Employing Eqs. (5)–(7) the reduction of the theoretical error in P_c from $\pm 10.1\%$ down to $\pm 2.4\%$ translates into the following uncertainties:

$$\frac{\sigma(|V_{td}|)}{|V_{td}|} = \begin{cases} \pm 4.1\%, & \text{NLO}, \\ \pm 1.0\%, & \text{NNLO}, \end{cases} \tag{10}$$

$$\sigma(\sin 2\beta) = \begin{cases} \pm 0.025, & \text{NLO}, \\ \pm 0.006, & \text{NNLO}, \end{cases} \tag{11}$$

$$\sigma(\gamma) = \begin{cases} \pm 4.9^\circ, & \text{NLO}, \\ \pm 1.2^\circ, & \text{NNLO}, \end{cases} \tag{12}$$

implying a very significant improvement of the NNLO over the NLO results. In obtaining these numbers we have used $\sin 2\beta = 0.724$ and $\gamma = 58.6^\circ$ [10], and included only the theoretical errors quoted in Eqs. (3) and (8).

On the experimental side the Alternating Gradient Synchrotron E787 and E949 Collaborations at Brookhaven observed the decay $K^+ \to \pi^+ \nu\bar{\nu}$ finding three events so far [25]. The resulting branching ratio is

$$\mathcal{B}(K^+ \to \pi^+ \nu\bar{\nu}) = (14.7^{+13.0}_{-8.9}) \times 10^{-11}. \tag{13}$$

Within theoretical, parametric, and experimental uncertainties, Eq. (9) is fully consistent with the data. The prospects for the future measurements of $\mathcal{B}(K^+ \to \pi^+ \nu\bar{\nu})$ can be found in [26].

To conclude, we have evaluated the complete NNLO correction of the charm quark contribution to $\mathcal{B}(K^+ \to \pi^+ \nu\bar{\nu})$. The inclusion of these contributions leads to a drastic reduction of the theoretical uncertainty in the relevant parameter P_c. This strengthens the power of the $K \to \pi\nu\bar{\nu}$ system in determining the CKM parameters and increases its reach to new physics, in particular, if future experimental values of $\mathcal{B}(K^+ \to \pi^+ \nu\bar{\nu})$ will not differ much from the SM prediction.

We would like to thank G. Buchalla, A. Höcker, M. Pierini, F. Schwab, and L. Silvestrini for discussions. A. J. B. is supported by the BMBF under Contract No. 05HT4WOA/3. U. H. and U. N. are supported by the DOE under Contract No. DE-AC02-76CH03000.

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