The Gravity Lagrangian According to Solar System Experiments

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In this work we show that the gravity Lagrangian $f(R)$ at relatively low curvatures in both metric and Palatini formalisms is a bounded function that can only depart from the linearity within the limits defined by well-known functions. We obtain those functions by analyzing a set of inequalities that any $f(R)$ theory must satisfy in order to be compatible with laboratory and solar system observational constraints. This result implies that the recently suggested $f(R)$ gravity theories with nonlinear terms that dominate at low curvatures are incompatible with observations and, therefore, cannot represent a valid mechanism to justify the cosmic speedup.

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Observations carried out in the last few years indicate that the Universe is undergoing a period of accelerated expansion [1]. In the context of this cosmic speedup, modified theories of gravity in which the gravity Lagrangian is a nonlinear function of the scalar curvature

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}f(R) + S_m[g_{\mu\nu}, \psi_m] \tag{1}$$

have become the object of recent interest. These theories are commonly referred to as $f(R)$ gravities. Motivated by the fact that the addition of positive powers of the scalar curvature to the Hilbert-Einstein Lagrangian may give rise to early-time inflationary periods, modifications of the Lagrangian that become dominant at low curvatures have been suggested to justify the observed late-time cosmic acceleration [2] (see also [3]). The aim of these theories is to describe the cosmic acceleration as an effect of the gravitational dynamics itself rather than as due to the existence of sources of dark energy. In addition to the standard formulation of $f(R)$ gravities, where the metric is the only gravitational field, it was pointed out in [4] that once nonlinear terms are introduced in the Lagrangian, an alternative and inequivalent formulation of these theories is possible. If one considers metric and connection as independent fields, i.e., that the connection in the gravity Lagrangian is not the usual Levi-Cività connection but is to be determined by the equations of motion, then the resulting theory is different from that defined in terms of the metric only. This new approach, known as Palatini formalism, and the metric one have been shown to lead to late-time cosmic acceleration for many different $f(R)$ Lagrangians. However, the attempts made so far to unravel the form of the function $f(R)$ from the cosmological data are far from being conclusive [5].

In this work we study the constraints on $f(R)$ gravities imposed by laboratory and solar system experiments. We find that the Lagrangian must be linear in R and that the possible nonlinear corrections are bounded by R^2. This implies that the cosmic speedup cannot be due to unexpected gravitational effects at low cosmic curvatures.

In order to confront the predictions of a given gravity theory with experiment in the solar system, it is necessary to obtain its weak field, slow motion (or post-Newtonian) limit. We will now derive a scalar-tensor representation for $f(R)$ gravities that will allow us to treat the metric and Palatini formulations in a very similar manner and will simplify the computations of the post-Newtonian metric. The following action (see [6,7]) leads to the same equations of motion as Eq. (1):

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}[f(A) + (R - A)B] + S_m, \tag{2}$$

where A is an auxiliary field and $B \equiv df(A)/dA$. To fix our notation, we will denote $R(g)$ as the contraction $R(g) \equiv g^{\mu\nu}R_{\mu\nu}$ with the Ricci tensor $R_{\mu\nu}$ given in terms of the Levi-Cività connection of $g_{\mu\nu}$, and $R(\Gamma)$ as the contraction $R(\Gamma) \equiv g^{\mu\nu}R_{\mu\nu}$ with $R_{\mu\nu}$ given in terms of a connection Γ independent of $g_{\mu\nu}$. Thus, the symbol R in Eq. (2) must be seen as $R = R(g)$ in the metric formalism, and as $R = R(\Gamma)$ in the Palatini formalism. By inverting the function $B = B(A)$ to get $A = A(B)$ and defining

$$V(B) = AB - f(A), \tag{3}$$

then Eq. (2) in the metric formalism can be identified with the case $\omega = 0$ of the Brans-Dicke-like theories,

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[\phi R(g) - \frac{\omega}{\phi}(\partial_\mu\phi\partial^\mu\phi) - V(\phi)\right] + S_m, \tag{4}$$

where $\phi \equiv B$. We thus see that B is the relevant field instead of $A = A(B)$. Analogously, one can find a scalar-tensor representation for the Palatini formulation. In this case, the equations of motion for the connection lead to

$$\Gamma^\alpha_{\beta\gamma} = \frac{t^{\alpha\lambda}}{2}(\partial_\beta t_{\lambda\gamma} + \partial_\gamma t_{\lambda\beta} - \partial_\lambda t_{\beta\gamma}), \tag{5}$$

where the tensor $t_{\mu\nu}$ is defined as $t_{\mu\nu} = Bg_{\mu\nu}$. This solution for the connection allows us to write $R(\Gamma)$ in terms of the metric and the field B as follows:

$$R(\Gamma) = R(g) + \frac{3}{2B}\partial_\lambda B \partial^\lambda B - \frac{3}{B}\Box B \tag{6}$$

Inserting this solution in Eq. (2) and discarding a total divergence we get

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[BR(g) + \frac{3}{2B}\partial_\mu B\partial^\mu B - V(B)\right] + S_m, \tag{7}$$

which represents the case $\omega = -3/2$ of the theories defined in Eq. (4). [This result can also be obtained from Eq. (24) of [7] by means of a conformal transformation back to the original $f(R)$ frame.] It is worth noting that when $V(B)$ is given, the inverse problem of finding the Lagrangian $f(R)$ is also possible. From Eq. (3) we see that

$$\frac{dV(B)}{dB} = A. \tag{8}$$

Using this algebraic equation to solve for $B = B(A)$, the Lagrangian can be written, using again Eq. (3), as

$$f(A) = AB - V(B). \tag{9}$$

This means that, with regard to the equations of motion, $f(R)$ gravities in metric and Palatini formalisms are equivalent to $\omega = 0$ and $\omega = -3/2$ Brans-Dicke-like theories, respectively.

The identification of $f(R)$ gravities with particular cases of Brans-Dicke-like theories should, in principle, allow us to express their post-Newtonian limit in a compact form dependent on the parameter ω. However, this is only partially true. A glance at the equation of motion for the scalar field defined in Eq. (4),

$$[3 + 2\omega]\Box\phi + 2V(\phi) - \phi V'(\phi) = \kappa^2 T, \tag{10}$$

where $T \equiv g^{\mu\nu}T_{\mu\nu}$, indicates that the field is a dynamical object for all ω except $\omega = -3/2$. Thus, there exists a clear dynamical difference between a generic $\omega =$ constant theory and the case $\omega = -3/2$, which corresponds to the Palatini form of $f(R)$ gravities. As a scalar-tensor theory, the case $\omega = -3/2$ seems to have been almost avoided in the literature. In the original Brans-Dicke theory, where the potential term was not present, this case was obviously pathological (see [8] for a discussion of the limit $\omega \to -3/2$). Moreover, it was found that in order to get agreement between predictions and solar system experiments ω should be large and positive, and little attention was paid later, when nontrivial potentials were considered, to small or negative values of ω. On the other hand, for $\omega \neq -3/2$ theories with $V \neq 0$ it is well known that if the field has associated a large effective mass, the predictions of the theory may agree with solar system experiments irrespective of the value of ω [9]. However, such a result assumes that the field is near an extremum of its potential. For $\omega = 0$ [$f(R)$ in metric formalism] that condition, $dV/d\phi = 0$, cannot be imposed in general. This follows from Eq. (8) and the A equation of motion, which lead to $dV/d\phi = R(g)$. Since the leading order of $R(g)$ at a given time coincides with the background cosmic curvature $R_0 \neq 0$, we cannot impose $dV/d\phi = 0$ for $\omega = 0$ theories. We are thus forced to compute the post-Newtonian limit of this case without making any *a priori* assumption or simplification about the behavior of the potential. In the case $\omega = -3/2$ the relation between ϕ and $V(\phi)$ [see Eq. (10)] is even stronger and also forces us not to make any assumption about $V(\phi)$.

We shall now sketch the basic steps to compute the post-Newtonian metric of Brans-Dicke-like theories, which will be detailed elsewhere. We generalize the results of the literature so as to include all the terms that are relevant for our discussion. For approximately static solutions, corresponding to masses such as the Sun or Earth, to lowest order, we can drop the terms involving time derivatives from the equations of motion. In a coordinate system in free fall with respect to the surrounding cosmological model, the metric can be expanded about its Minkowskian value as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.

For $\omega \neq -3/2$, the field is dynamical and can be expanded as $\phi = \phi_0 + \varphi(t, x)$, where $\phi_0 \equiv \phi(t_0)$ is the asymptotic cosmic value, which is a slowly varying function of the cosmic time t_0, and $\varphi(t, x)$ represents the local deviation from ϕ_0. To lowest order ($T \approx -\rho$), the metric satisfies the following equations:

$$-\frac{1}{2}\nabla^2\left[h_{00} - \frac{\varphi}{\phi_0}\right] = \frac{\kappa^2\rho}{2\phi_0} - \frac{V_0}{2\phi_0}, \tag{11}$$

$$-\frac{1}{2}\nabla^2\left[h_{ij} + \delta_{ij}\frac{\varphi}{\phi_0}\right] = \delta_{ij}\left[\frac{\kappa^2\rho}{2\phi_0} + \frac{V_0}{2\phi_0}\right], \tag{12}$$

where the gauge condition $h^\mu_{k,\mu} - \frac{1}{2}h^\mu_{\mu,k} = \partial_k\varphi^{(2)}/\phi_0$ has been used. In eliminating the zeroth-order terms in the field equation for φ, corresponding to the cosmological solution for ϕ_0, we obtain

$$[\nabla^2 - m_\varphi^2]\varphi^{(2)}(t, x) = -\frac{\kappa^2\rho}{3 + 2\omega}, \tag{13}$$

where m_φ^2 is a slowly varying function of the cosmic time

$$m_\varphi^2 \equiv \frac{\phi_0 V_0'' - V_0'}{3 + 2\omega}. \tag{14}$$

Solving Eqs. (11)–(13), the metric becomes

$$h_{00} = 2G\frac{M_\odot}{r} + \frac{V_0}{6\phi_0}r^2, \tag{15}$$

$$h_{ij} = \delta_{ij}\left[2\gamma G\frac{M_\odot}{r} - \frac{V_0}{6\phi_0}r^2\right], \tag{16}$$

where $M_\odot = \int d^3x'\rho(t, x')$, and we have defined the effective Newton's constant and the parametrized post-Newtonian parameter γ as

$$G = \frac{\kappa^2}{8\pi\phi_0}\left(1 + \frac{F(r)}{3+2\omega}\right), \quad (17)$$

$$\gamma = \frac{3+2\omega - F(r)}{3+2\omega+F(r)}. \quad (18)$$

The function $F(r)$ is given by $F(r) = e^{-|m_\varphi|r}$ when $m_\varphi^2 > 0$, and by $F(r) = \cos|m_\varphi|r$ when $m_\varphi^2 < 0$.

Let us now obtain the metric for $\omega = -3/2$ theories. In this case, no boundary conditions are needed for the scalar field, since it satisfies an algebraic equation. Denoting by $\phi = \phi(T)$ the solution to Eq. (10) when $\omega = -3/2$, we construct the quantity $\Omega(T) \equiv \log[\phi/\phi_0]$, where the subindex now denotes vacuum value, $\phi_0 \equiv \phi(T=0)$. Using the gauge condition $h^\mu_{k,\mu} - \frac{1}{2}h^\mu_{\mu,k} = \partial_k\Omega(T)$, the equations for the metric can be written as

$$-\frac{1}{2}\nabla^2[h_{00} - \Omega(T)] = \frac{\kappa^2\rho - V(\phi)}{2\phi}, \quad (19)$$

$$-\frac{1}{2}\nabla^2[h_{ij} + \delta_{ij}\Omega(T)] = \left[\frac{\kappa^2\rho + V(\phi)}{2\phi}\right]\delta_{ij}. \quad (20)$$

The solution to these equations are

$$h_{00}(t,x) = 2G\frac{M_\odot}{r} + \frac{V_0}{6\phi_0}r^2 + \Omega(T), \quad (21)$$

$$h_{ij}(t,x) = \left[2\gamma G\frac{M_\odot}{r} - \frac{V_0}{6\phi_0}r^2 - \Omega(T)\right]\delta_{ij}, \quad (22)$$

where $M_\odot \equiv \phi_0 \int d^3x'\rho(t,x')/\phi$, and we have defined

$$G = \frac{\kappa^2}{8\pi\phi_0}\left(1 + \frac{M_V}{M_\odot}\right), \quad (23)$$

$$\gamma = \frac{M_\odot - M_V}{M_\odot + M_V}, \quad (24)$$

with $M_V \equiv \kappa^{-2}\phi_0 \int d^3x'[V_0/\phi_0 - V(\phi)/\phi]$.

The term $(V_0/\phi_0)r^2$ appearing in Eqs. (15) and (16) for $\omega \neq -3/2$ and in (21) and (22) for $\omega = -3/2$ is related to the scalar energy density and acts in a manner similar to the effects of a cosmological constant. Thus, any viable theory must give a negligible contribution of this type. Let us now focus on $\omega = 0$ theories. The oscillating solutions $F(r) = \cos|m_\varphi|r$ are always unphysical. If $|m_\varphi|L \ll 1$ with L large compared to solar system scales (long-range interaction), we find $\gamma \approx 1/2$, which is ruled out by observations ($\gamma_{\rm obs} \approx 1$; see [10]). If $|m_\varphi|L \gg 1$ (short range), Newton's constant strongly oscillates in space and the Newtonian limit is dramatically modified. Thus, only the damped case $F(r) = e^{-|m_\varphi|r}$ with

$$m_\varphi^2 L^2 \gg 1 \quad (25)$$

is physically acceptable. This condition means that the scalar interaction range $l_\varphi = m_\varphi^{-1}$ must be shorter than any currently accessible experimental length L.

In the case $\omega = -3/2$, we can learn about the dependence of ϕ on T by studying the behavior of $M_\odot$, G, and γ. According to the definition of $M_\odot$ and M_V [see definitions following Eqs. (22) and (24)], it follows that a body with Newtonian mass $M_N \equiv \int d^3x'\rho(t,x')$ may yield different values of $M_\odot$, G, and γ depending on its internal structure and composition. Consequently, a given amount of Newtonian mass could lead to gravitational fields of different strengths and dynamical properties. We thus demand a very weak dependence of ϕ on T so as to guarantee that $M_\odot$, G, and γ are almost constant. This is consistent with the requirement that the local term $\Omega(T)$ in Eqs. (21) and (22) must be small compared to unity. Since the contribution of $\Omega(T)$ to the acceleration of a body is given in terms of its gradient, we must demand that

$$\left|\frac{T(\partial\phi/\partial T)}{\phi}\right| \ll 1 \quad (26)$$

from $T = 0$ up to nuclear densities. Otherwise, a change in ϕ when going from its vacuum value ϕ_0 outside atoms to its value inside atoms would lead to observable effects in the motion of macroscopic test bodies placed in the gravitational field defined by Eqs. (21) and (22). Since such effects have not been observed, it follows that over a wide range of densities $\phi \approx \phi_0 + (\partial\phi/\partial T)|_{T=0}T$, with $|\phi_0^{-1}(\partial\phi/\partial T)|_{T=0}T| \ll 1$, must be a very good approximation. The weak dependence on T expressed by Eq. (26) can be written using Eq. (10) to evaluate $(\partial\phi/\partial T)$ as

$$\left|\frac{(\kappa^2\rho/\phi)}{(\phi V'' - V')}\right| \ll 1. \quad (27)$$

We can thus interpret this equation in a manner analogous to Eq. (25), as the ratio of a length associated with the matter density, $\mathcal{L}^2(\rho) = [\kappa^2\rho/\phi_0]^{-1}$, over a length associated with the scalar field, $l_\phi^2 = m_\phi^{-2} \equiv [(\phi V'' - V')\phi/\phi_0]^{-1}$.

The constraint on $\omega = 0$ theories given in Eq. (25) can be rewritten in terms of the Lagrangian $f(R)$ as follows:

$$R_0\left[\frac{f'(R_0)}{R_0 f''(R_0)} - 1\right]L^2 \gg 1, \quad (28)$$

where R_0 represents the current cosmic scalar curvature. Note that since $\phi \equiv f' > 0$ to have a well posed theory, it follows that f'' must be small and positive in order to satisfy Eq. (28). Consequently, any theory with $f'' < 0$ leads to an ill defined post-Newtonian limit ($m_\varphi^2 < 0$). This is the case, for instance, of the Carroll *et al.* model [2]. We shall now demand that the interaction range of the scalar field remains as short as it is today or decreases with time so as to avoid dramatic modifications of the gravitational dynamics in post-Newtonian systems with the cosmic expansion. This can be implemented imposing

$$\left[\frac{f'(R)}{Rf''(R)} - 1\right] \geq \frac{1}{l^2 R} \quad (29)$$

as $R \to 0$, where $l^2 \ll L^2$ represents a bound to the current interaction range of the scalar field. Manipulating this inequality, we obtain

$$\frac{d\log[f'(R)]}{dR} \leq \frac{l^2}{1 + l^2 R}, \tag{30}$$

which can be integrated twice to give

$$f(R) \leq \alpha + \beta\left(R + \frac{l^2 R^2}{2}\right), \tag{31}$$

where $\beta > 0$. Since f' and f'' are positive, the Lagrangian is also bounded from below, say, $f(R) \geq \alpha$. According to the cosmological data, $\alpha \equiv -2\Lambda$ must be of order a cosmological constant $2\Lambda \sim 10^{-53}\ \mathrm{m}^2$. Without loss of generality, setting $\beta = 1$ we find that, in order to satisfy the current solar system constraints, the Lagrangian in metric formalism must satisfy

$$-2\Lambda \leq f(R) \leq R - 2\Lambda + \frac{l^2 R^2}{2}, \tag{32}$$

which is clearly incompatible with nonlinear terms growing at low curvatures.

Let us consider now the Palatini case, $\omega = -3/2$. Written in terms of the Lagrangian $f(R)$, Eq. (27) turns into

$$R\tilde{f}'(R)\left|\frac{\tilde{f}'(R)}{R\tilde{f}''(R)} - 1\right| \mathcal{L}^2(\rho) \gg 1, \tag{33}$$

where $\tilde{f}' \equiv f'/f'_0 = \phi/\phi_0$. Since $\mathcal{L}^2(\rho) \sim 1/\rho$ takes its smallest value for ordinary matter at nuclear densities, it is reasonable to demand that

$$\left|\frac{\tilde{f}'(R)}{R\tilde{f}''(R)} - 1\right| \geq \frac{1}{l^2 R\tilde{f}'}, \tag{34}$$

where l^2 represents a length scale much smaller than $\mathcal{L}^2(\rho)$ at those densities. Note that l^2 determines the scale over which the nonlinear corrections are relevant. If $l^2 = 0$, which implies $f(R) = a + bR$, then Eq. (33) would be valid for all ρ. Furthermore, if the nonlinear corrections were important at very low cosmic densities, l would be of order the radius of the Universe and the nonlinear terms would dominate at all scales, which would lead to unacceptable predictions, as we pointed out above with regard to $M_\odot$, G, γ, and $\Omega(T)$. A good example of such pathological effects in the Palatini formalism was studied in [7]. Manipulating Eq. (34), we obtain for $\tilde{f}'' > 0$

$$f \leq a + \frac{l^2 R^2}{2} + \frac{R}{2}\sqrt{1 + (l^2 R)^2} + \frac{1}{2l^2}\log[l^2 R + \sqrt{1 + (l^2 R)^2}] \tag{35}$$

and for $\tilde{f}'' < 0$

$$f \geq a - \frac{l^2 R^2}{2} + \frac{R}{2}\sqrt{1 + (l^2 R)^2} + \frac{1}{2l^2}\log[l^2 R + \sqrt{1 + (l^2 R)^2}], \tag{36}$$

where the vacuum value f'_0 has been set to unity and a can be identified with $a \equiv -2\Lambda$. We see that, to leading order in $l^2 R$, the Palatini Lagrangian is bounded by

$$R - 2\Lambda - \frac{l^2 R^2}{2} \leq f(R) \leq R - 2\Lambda + \frac{l^2 R^2}{2}. \tag{37}$$

According to Eqs. (32) and (37), our conclusion is clear: laboratory and solar system experiments indicate that the gravity Lagrangian is nearly linear in R, with the possible nonlinearities bounded by quadratic terms. Consequently, $f(R)$ Lagrangians with nonlinear terms that grow at low curvatures cannot represent a valid mechanism to justify the cosmic accelerated expansion rate. Such theories lead to a long-range scalar interaction incompatible with the experimental tests. In the viable models the nonlinearities represent a short-range scalar interaction, whose effect in the late-time cosmic dynamics reduces to that of a cosmological constant and, therefore, do not substantially modify the description provided by general relativity. To conclude, we want to remark that Eqs. (32), (35), and (36) tell us how the Lagrangian $f(R)$ must be near the origin, $l^2 R \ll 1$, not far from it, $l^2 R \gg 1$, where the post-Newtonian constraints could not make sense. This fact constraints the possible $f(R)$ early-time inflationary models. In particular, $f(R) = R + aR^2$ in metric formalism seems compatible with cosmic microwave background radiation observations [11].

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