Logarithmic Conformal Field Theory and Boundary Effects in the Dimer Model

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(Received 25 July 2005; published 19 December 2005)

We study the finite-size corrections of the dimer model on a $\infty \times N$ square lattice with two different boundary conditions: free and periodic. We find that the finite-size corrections depend in a crucial way on the parity of N; we also show that such unusual finite-size behavior can be fully explained in the framework of the $c = -2$ logarithmic conformal field theory.

DOI: 10.1103/PhysRevLett.95.260602 PACS numbers: 05.50.+q, 05.70.Fh, 75.10.−b

Universality, scaling, and exact finite-size corrections in critical systems have attracted much attention in recent decades [1–4]. It has been found that critical systems can be classified into different universality classes so that the systems in the same class have the same set of critical exponents, universal finite-size scaling functions, and amplitude ratios [1,2]. Two-dimensional critical systems are parametrized by the central charge c [5], directly related to the finite-size corrections to the critical free energy [6].

In this Letter, we address this question for dimers defined on a square lattice, with three main purposes: (i) We dissipate the confusion existing in the literature about the value of the central charge [7,8] due to the (mis)interpretation of the finite-size corrections in terms of the central charge rather than the effective central charge; (ii) we give a bijection of dimer coverings with a spanning tree and Abelian sandpile model, which not only allows a proper understanding of the dimer model but proves also very useful to calculate finer effects, such as the change of boundary conditions; (iii) using this bijection, we clarify and explain why a change of parity of the lattice size causes a change of the effective central charge but not of the central charge itself.

The dimer problem on planar lattices belongs to the class of "free-fermion" models [9]. Its solution has been obtained with the Pfaffian approach [10] and then reproduced by a variety of methods [11]. In contrast to the statistics of simple particles, the critical behavior of the dimer model is strongly influenced by the structure of the lattice space. The square lattice dimer model is critical with algebraic decay of correlators [12]. For the dimer model on the anisotropic honeycomb lattice, which is equivalent to a five-vertex model on the square lattice [13], the free energy exhibit a potassium dihydrogen phosphate-type singularity. For the triangular and some decorated lattices, the dimer model exhibits Ising-type transitions [14]. Thus, it appears that the dimer model itself has not a single critical behavior but several critical behaviors associated with different universality classes.

In what follows, we consider the finite-size effects for close-packed dimers on finite square lattices, with free boundary conditions on all sides (strip geometry) and with a periodic boundary condition in one direction (cylinder). In all cases, we find them to be consistent with a central charge $c = -2$. This conclusion relies on a careful distinction between the central charge c and the so-called effective central charge $c_{\rm eff} = c - 24h_{\rm min}$ [15], which is a boundary dependent quantity. The value $c = -2$ is further confirmed by calculating the effect of a change of boundary conditions. We find that, in the scaling limit, it corresponds to the insertion of a boundary primary field of weight $-1/8$, belonging to a logarithmic conformal field theory with $c = -2$.

Finite-size analysis.—Let us consider the dimer model on an $M \times N$ square lattice $\mathcal{L}$ with M rows and N columns. The topology of $\mathcal{L}$ is fixed by the boundary conditions: It forms a rectangle if free boundary conditions are imposed in two directions, a cylinder or a torus if periodic boundary conditions are chosen in only a horizontal direction or two directions.

The partition function of the dimer model is given by

$$Z_{M,N}(z_v, z_h) = \sum z_v^{n_v} z_h^{n_h}, \tag{1}$$

where the summation is over all dimer covering configurations, z_v and z_h are the dimer weights in the vertical and horizontal directions, respectively, and n_v and n_h are the numbers of vertical and horizontal dimers, respectively.

The partition functions of the dimer model with the boundary conditions discussed above can all be expressed in terms of $Z_{\alpha,\beta}(z, M, N)$ for $\alpha, \beta = 0, \frac{1}{2}$ [4] with

$$Z^2_{\alpha,\beta}(z, M, N) = \prod_{n=0}^{N-1}\prod_{m=0}^{M-1} 4\left[z^2 \sin^2\frac{\pi(n+\alpha)}{N} + \sin^2\frac{\pi(m+\beta)}{M}\right]. \tag{2}$$

Here $z = z_h/z_v$, which we set equal to 1 from now on.

The general theory about the asymptotic expansion of $Z_{\alpha,\beta}$ for large M, N has been presented in Ref. [3]. For what follows, the asymptotic expansion of the free energy per unit length associated to $Z_{\alpha,\beta}$ is all we need. The result reads [3]

$$\begin{aligned} F_{\alpha,\beta}(N) &= -\lim_{M\to\infty}\frac{1}{M}\ln Z_{\alpha,\beta}(1,M,N) \\ &= -\frac{2G}{\pi}N + \sum_{p=0}^{\infty}\left(\frac{\pi}{N}\right)^{2p+1}\frac{2z_{2p}}{(2p)!}\frac{B_{2p+2}(\alpha)}{2p+2}, \end{aligned} \tag{3}$$

where $z_0 = 1$, $z_2 = -2/3, \ldots$; $G = 0.915\,965$ is the Catalan constant, and the $B_p(\alpha)$ are the Bernoulli polynomials, $B_2(\alpha) = \alpha^2 - \alpha + \frac{1}{6}$. These formulas allow one to compute the asymptotic expansion of the free energy F_N per unit length of the $\infty \times N$ lattice for large N and for free and periodic boundary conditions.

Let us first consider the case of an infinitely long strip of width N with free boundary conditions. For N even, the results of Ref. [4] show that

$$F^{\text{free}}_{N,\text{even}} = \tfrac{1}{2}F_{1/2,0}(N+1) + \log(1+\sqrt{2}). \tag{4}$$

The expansion (3) then gives

$$F^{\text{free}}_{N,\text{even}} = -\frac{G}{\pi}(N+1) + \log(1+\sqrt{2}) - \frac{\pi}{24}\frac{1}{N} + \cdots. \tag{5}$$

Similar calculations for N odd lead to a formula in terms of $Z_{0,1/2}(N+1)$ and yield

$$F^{\text{free}}_{N,\text{odd}} = -\frac{G}{\pi}(N+1) + \log(1+\sqrt{2}) + \frac{\pi}{12}\frac{1}{N} + \cdots. \tag{6}$$

The analogous results for the periodic case, i.e., an infinite cylinder of perimeter N, read

$$F^{\text{per}}_{N,\text{even}} = -\frac{G}{\pi}N - \frac{\pi}{6}\frac{1}{N} + \cdots, \tag{7}$$

$$F^{\text{per}}_{N,\text{odd}} = -\frac{G}{\pi}N + \frac{\pi}{12}\frac{1}{N} + \cdots. \tag{8}$$

The free energy per unit length of an infinitely long strip of finite width N at criticality has the finite-size scaling form [6]

$$F = f_{\text{bulk}}N + f_{\text{surf}} + \frac{A}{N} + \cdots, \tag{9}$$

where f_{bulk} and f_{surf} are, respectively, the bulk and the surface (boundary) free energy densities, and A is a constant. Though the free energy densities f_{bulk} and f_{surf} are not universal, the constant A is universal. The value of A is related to the conformal anomaly c of the underlying conformal theory and depends on the boundary conditions in the transversal direction. These two dependencies combine into a function of the effective central charge $c_{\text{eff}} = c - 24h_{\min}$,

$$A = -\frac{\pi}{24}c_{\text{eff}} = \pi\left(h_{\min} - \frac{c}{24}\right) \text{ on a strip,} \tag{10}$$

$$A = -\frac{\pi}{6}c_{\text{eff}} = 4\pi\left(h_{\min} - \frac{c}{24}\right) \text{ on a cylinder.} \tag{11}$$

The number $h_{\min}$ is the (chiral) conformal weight of the operator with the smallest scaling dimension present in the spectrum of the Hamiltonian with the given boundary conditions (for the cylinder, we assumed that this operator is scalar, $h_{\min} = \bar{h}_{\min}$).

In a unitary theory, one has $h_{\min} = 0$ on a cylinder (with a periodic condition) and on a strip with identical left and right boundary conditions, and $h_{\min} > 0$ otherwise. In a nonunitary theory, such as the conformal theory discussed here, there is no restriction on $h_{\min}$.

The finite-size corrections computed above have all the form (9), with the effective central charge depending on the parity of N; see (10) and (11). We will show that indeed the effective central charge, and not the central charge itself, depends on the parity of N, because the value of $h_{\min}$ does, due to the fact that changing the parity of N in effect changes the boundary condition.

To understand this peculiarity of the dimer model, we consider, first on the strip and then on a cylinder, the mapping of the dimer model to the spanning tree model [16] and, equivalently, the Abelian sandpile model [17].

Dimers on a strip.—Let us consider first the dimer model on the rectangular lattice $\mathcal{L}$ of size $M \times N$ with free boundary conditions. Since we are interested in the limit $M \to \infty$, the parity of M will not matter here. For simplicity, we take M odd and discuss successively the cases N odd and N even.

For M and N both odd, the bijection between dimer coverings on $\mathcal{L}$ with one corner removed and spanning trees on the odd sublattice $\mathcal{G} \subset \mathcal{L}$ is well known [7,16].

A dimer containing a site of $\mathcal{G}$, in blue in Fig. 1, can be represented as an arrow directed along the dimer from this site to the nearest neighbor site of $\mathcal{G}$. It is easy to prove that the resulting set of arrows generates a uniquely defined spanning tree, rooted at the corner which had been removed from $\mathcal{L}$ (see Fig. 1). Since the dimers which do not contain a site of $\mathcal{G}$ are completely fixed by the others, one has a one-to-one correspondence between dimer coverings on $\mathcal{L}$ minus a corner and spanning trees on $\mathcal{G}$. This allows one to express the number of dimer configurations by the Kirchhoff theorem as $Z = \det\Delta$, where Δ is the Laplacian matrix for spanning trees on $\mathcal{G}$. As shown in Ref. [17], spanning trees on $\mathcal{G}$, rooted at a corner, are in bijection with the configurations of the Abelian sandpile model (ASM) on $\mathcal{G}$, with closed boundary conditions on the four boundaries, the only sink (dissipative) site being the root of the trees.

When $M \to \infty$, the lattice $\mathcal{G}$ becomes an infinitely long strip of width N and, in the ASM language, closed boundary conditions on the two vertical sides. Many independent arguments and explicit calculations all converge to a central charge $c = -2$ for the ASM on a square lattice [17–19]. On the other hand, the spectrum of the ASM Hamiltonian on a slice of the strip with closed boundary

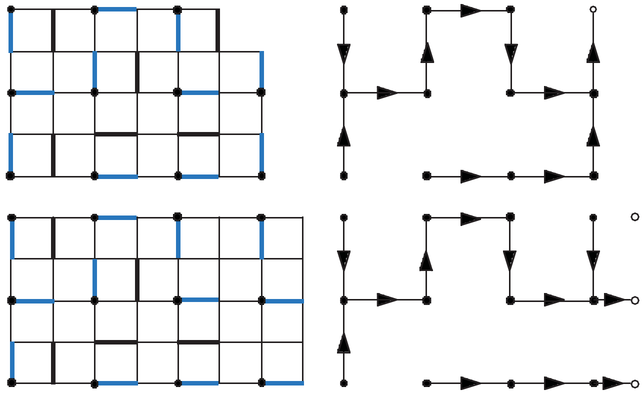

FIG. 1 (color). Mapping of a dimer covering to a spanning tree on the odd sublattice, for an $M \times N = 5 \times 7$ lattice (top) and a 5×8 lattice (bottom). In both cases, the solid dots represent the sites of the odd sublattice $\mathcal{G}$, and the open dots are the roots of the trees.

conditions at the two ends has been computed in Ref. [18]. It is given by a single representation $\mathcal{R}$, reducible but indecomposable, of the chiral algebra of a rational logarithmic conformal field theory with $c = -2$ [20]. The representation has two ground states, the identity operator and its logarithmic partner, both of conformal weight 0, so that $h_{\min} = 0$. The effective central charge in this sector is, therefore, $c_{\text{eff}} = -2$, and the general formula (10) reproduces the finite-size corrections (6).

When M is odd and N is even, dimer coverings exist on $\mathcal{L}$ without the need to remove a corner. In this case, the above construction leads to a set of spanning trees on the odd sublattice $\mathcal{G}$, where certain arrows may point out of the lattice from the right vertical side (Fig. 1). Viewing this vertical boundary of $\mathcal{G}$ as roots for the spanning trees, we see that dimer coverings on $\mathcal{L}$ map onto spanning trees on $\mathcal{G}$ which can grow from any site of the vertical side. In turn, the spanning trees map onto the ASM configurations with one vertical open, dissipative boundary and the three other closed.

In the limit $M \to \infty$, the lattice becomes an infinite strip with open and closed boundary conditions on the two sides. In this case, the results of Ref. [18] show that the ground state of the Hamiltonian with such boundary conditions is a primary field of conformal weight $h_{\min} = -1/8$. With $c = -2$, this yields $c_{\text{eff}} = 1$ and again the formula (10) correctly gives the result (5).

Let us note that the bijection between the dimer coverings and the spanning trees holds if we use the even sublattice instead of the odd one. The boundary conditions, however, change. If N is odd, the vertical sides (and the horizontal ones as well, for M odd) become open rather than closed. The spectrum of the corresponding Hamiltonian change, with a nondegenerate ground state being the identity operator [18]. Thus, the value $h_{\min} = 0$ remains. If N is even, the left and right boundaries, previously closed and open, respectively, become open and closed, so that the Hamiltonian remains the same, $h_{\min} = -1/8$.

The equivalence of the odd and even sublattices is a duality property. The two sublattices are dual to each other, and the spanning trees on $\mathcal{G}_{\text{even}}$ are dual to those on $\mathcal{G}_{\text{odd}}$. It is not difficult to check that open and closed boundary conditions are exchanged under duality.

Thus, the leading finite-size corrections for an infinitely long strip of width N agree with the prediction of a $c = -2$ conformal field theory, provided one realizes that changing the parity of N genuinely changes the boundary conditions, an effect due to the strong nonlocality of the dimer model. The change of boundary conditions is not apparent in the dimer model itself but is manifest when one maps it onto the spanning tree model or the sandpile model.

The primary field with $h = -1/8$ can be further tested [18]. If, on a rectangular lattice $M \times N$, we remove n sites from the lower boundary, the height will be M or $M - 1$ and will, therefore, take on the two parities. This has the effect of changing the boundary condition along the lower boundary, from closed to open, and back to closed (or vice versa). We have checked that the universal part of the ratio of partition functions, after and before the removal, and expected to be equal to $\langle \phi_h(0)\phi_h(n)\rangle \sim n^{-2h}$, where ϕ_h is the boundary field that changes a boundary condition from open to closed, is indeed asymptotically equal to $n^{1/4}$, in the limit $M, N \to \infty$, supporting $h = -1/8$.

Dimers on a cylinder.—We consider here an $M \times N$ rectangular lattice $\mathcal{L}$ with a periodic boundary condition in the horizontal direction, so that $\mathcal{L}$ is a cylinder of perimeter N and height M. As before, we will eventually take M to infinity, which makes its parity irrelevant. We choose M even. According to the discussion of the previous section, the top boundary of the cylinder then is subjected to open boundary conditions (in ASM terms) while the bottom one is closed. We separate the cases N odd and N even. If N is odd, we select the sublattice $\mathcal{G}$ consisting of those sites of $\mathcal{L}$ having odd-odd coordinates. It is easy to see that two columns of $\mathcal{G}$ will contain sites which are neighbors in $\mathcal{G}$ and in $\mathcal{L}$ (connected by horizontal bonds). Therefore, a dimer may contain zero, one, or two sites of $\mathcal{G}$. The dimers containing no site of $\mathcal{G}$ are completely fixed by the others and play no role. For the others, we do the same construction as before. If a dimer touches one site of $\mathcal{G}$, we draw an arrow directed along the dimer from that site to the nearest neighboring site of $\mathcal{G}$. However, for a dimer containing two sites of $\mathcal{G}$, the two arrows would point from either site to the other, ruining the spanning tree picture. It can, nevertheless, be restored in the following way.

Instead of seeing the two arrows as pointing from one site to its neighbor, we say that they point towards roots inserted between the neighbor sites, thus replacing the arrows $\bullet\!\!\rightarrow\!\!\leftarrow\!\!\bullet$ by $\bullet\!\!\rightarrow\!\circ\,\circ\!\leftarrow\!\!\bullet$. This in effect amounts to cutting the cylinder along the bonds of $\mathcal{L}$ which connect sites of $\mathcal{G}$, unwrapping it into a strip, and adding columns of roots on the left and right sides of the strip. The new arrow configurations define spanning trees, rooted anywhere on the left and right boundaries. So dimer coverings on the original cylinder are mapped to spanning trees on a strip, with open top and closed bottom horizontal boundaries (we chose M even) and open vertical boundaries.

When M goes to infinity, the lattice becomes an infinite strip with an open boundary condition on either side. As mentioned above, the ground state of the Hamiltonian is the identity, of weight $h_{\min} = 0$, leading to an effective central charge $c_{\mathrm{eff}} = -2$. The general formula (10) for the strip gives the correct result (8).

This is a very unusual situation. Although the dimer model is originally defined on a cylinder, it shows the finite-size corrections expected on a strip and must really be viewed as a model on a strip.

For N even, the problem of having two arrows pointing from and to neighbor sites does not arise; however, the arrows one obtains do not define spanning trees but rather a combination of loops wrapped around the cylinder and tree branches attached to the loops. Each loop has two possible orientations. The one-to-one correspondence between the oriented loops combined with tree branches from one side and dimer configurations from the other side can be established as above. The enumeration of the loop-tree configurations needs a generalization of the Kirchhoff theorem, $Z = \det\tilde{\Delta}$, where $\tilde{\Delta}$ is the discrete Laplacian with antiperiodic boundary conditions. In the continuum limit, this leads to the free theory of antiperiodic Grassmann fields which, in turn, gives $h_{\min} = -1/8$ [21]. From the general formula (11) for the cylinder, we see that the finite-size correction (7) is again consistent with $c = -2$.

The above interpretation of the peculiarities of the dimer model shed a new light on old calculations by Ferdinand of the partition function of the dimer model on a $M \times N$ torus [22]. When M and N are both even, the universal part of the partition function equals

$$Z_{\mathrm{even,even}} = \frac{\theta_2^2 + \theta_3^2 + \theta_4^2}{2\eta^2}(q), \tag{12}$$

in the limit $M, N \to \infty$ with fixed ratio $q = \exp(-2\pi \frac{M}{N})$. This is exactly equal to the partition function $Z_{c=-2}(q) = |\chi_{-1/8}|^2 + 2|\chi_0 + \chi_1|^2 + |\chi_{3/8}|^2$ of the $c = -2$ rational conformal theory developed in Ref. [20], confirming the value of $c = -2$ (and the value $h_{\min} = -1/8$ found for the even cylinder). In case one of M or N is odd, the partition functions are

$$Z_{\mathrm{odd,even}} = \frac{\theta_2}{2\eta}(\sqrt{q}), \qquad Z_{\mathrm{even,odd}} = \frac{\theta_4}{2\eta}(\sqrt{q}). \tag{13}$$

These are cylinder partition functions of the same $c = -2$ conformal theory [18], in agreement with the view that a periodic dimension of odd size is actually not periodic, and correspondingly tori with one odd dimension are, in fact, cylinders.

We thank F. Y. Wu for valuable comments. V. P. and P. R. thank the hospitality of the Laboratory of Statistical and Computational Physics in Academia Sinica (Taiwan), where the most significant part of this work was done. This work was supported by National Science Council (Taiwan) under Grants No. NSC 93-2112-M001-027 and No. NSC 94-2119-M-002-001, Academia Sinica (Taiwan) under Grant No. AS-91-TP-A02, and by the Belgian Fonds National de la Recherche Scientifique.

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