Magnetic-Field Generation in Helical Turbulence

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We investigate analytically the amplification of a weak magnetic field in a homogeneous and isotropic turbulent flow lacking reflectional symmetry (helical turbulence). We propose that the spectral distributions of magnetic energy and magnetic helicity can be found as eigenmodes of a self-adjoint, Schrödinger-type system of evolution equations. We argue that large-scale and small-scale magnetic fluctuations cannot be effectively separated, and that the conventional α model is, in general, not an adequate description of the large-scale dynamo mechanism. As a consequence, the correct numerical modeling of such processes should resolve magnetic fluctuations down to the very small, resistive scales.

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Introduction.—It is well established both analytically and numerically that a weak magnetic field can be amplified by the random motions of a highly conducting fluid [1–3]. This occurs because magnetic-field lines are generically stretched by the random motions of the fluid in which they are (almost) "frozen." Such mechanisms of turbulent dynamo action are invoked to explain the origin of magnetic fields in astrophysical systems, such as planets, stars, the interstellar and the intergalactic medium, etc.

In many cases magnetic fields are observed to be strong and ordered on scales much larger than the velocity correlation length. The traditional view is that the origin of these fields can still be explained in the framework of isotropic and homogeneous turbulence provided the latter lacks reflectional symmetry. For this case the helicity integral of the velocity can be nonzero, i.e., $\mathcal{H} = \int \mathbf{v}\cdot(\nabla\times\mathbf{v})dV \neq 0$.

To illustrate this idea let us *assume* that the magnetic field, compared to the velocity correlation length, $l_{\rm vel}$, possesses only large-scale and small-scale components. The magnetic-field evolution is described by the induction equation

$$\partial_t \mathbf{B} = \nabla\times(\mathbf{v}\times\mathbf{B}) + \eta\Delta\mathbf{B}, \tag{1}$$

where η is the (collisional) diffusivity. Averaging over the small-scale fluctuations, $l \lesssim l_{\rm vel}$ we obtain the equation for the large scale, or the mean magnetic field $\bar{\mathbf{B}}(x, t)$. It can be written in the general form [3,4],

$$\partial_t \bar{\mathbf{B}} = \nabla\times(\alpha\bar{\mathbf{B}}) + \beta\Delta\bar{\mathbf{B}}, \tag{2}$$

where it is assumed that the mean field varies slowly in space, and, therefore, its higher-order spatial derivatives can be neglected. The parameters α and β can be estimated on dimensional grounds to be $\alpha \sim \langle\mathbf{v}\cdot(\nabla\times\mathbf{v})\rangle\tau_v$, and $\beta \sim \langle\mathbf{v}^2\rangle\tau_v$, where τ_v is the velocity correlation time.

In Fourier space, the linear Eq. (2) has the eigenvalues, $\lambda_1 = -\beta k^2$, and $\lambda_{2,3} = -\beta k^2 \pm \alpha k$, where k is the wave number. Thus a growing eigenmode always exists provided small enough wave numbers are allowed in the system under consideration (galaxy, laboratory device, simulation box). The maximal growth rate is then given by $\gamma_0 = \alpha^2/(4\beta)$, and the corresponding scale of the growing mean magnetic field is $l_0 \sim 2\beta/\alpha$. In order to comply with the underlying assumption of scale separation it is assumed that this scale is much larger than the velocity correlation scale $l_{\rm vel}$.

The mean-field growth rate γ_0 vanishes if the velocity fluctuations possess no helicity, $\mathcal{H} = 0$. One therefore might expect that the generation of magnetic fields at large scales, $l > l_{\rm vel}$, in homogeneous and isotropic turbulence may only be possible if the velocity field lacks reflectional symmetry, and that such magnetic fields are described by the mean-field equation (2). This effect is traditionally called the α-dynamo mechanism.

However, numerical results suggest that the large-scale magnetic-field evolution in helical turbulence may not be adequately described by the α mechanism (2). For example, Vainshtein and Cattaneo [5] noted that the small-scale magnetic fields are amplified more effectively than the large-scale ones, and when their energy is large enough to affect the velocity dynamics, the α mechanism may become much less effective. The influence of small-scale magnetic fields on the large-scale dynamo mechanism (2) has been stressed in many works (see, e.g., Refs. [6–8]).

Previous investigations of inconsistencies related to the α-dynamo mechanism (2) essentially concentrated on the nonlinear effects related to dynamo saturation, which, so far, have resisted exact analytical treatment. Present-day direct numerical simulations of turbulent dynamo action cannot provide conclusive results either, due to quite limited numerical resolution, e.g., Ref. [9].

In this Letter we propose that some of the essential physics of large-scale magnetic-field generation is, in fact, captured already at the initial, *kinematic* stage of dynamo action. Our analysis is based on the exactly solvable model of dynamo action due to Kazantsev [10]. Despite the many simplifying assumptions about the statistics of the velocity field this model has proven to be a valuable tool in understanding the dynamo mechanism. In particular, it treats the induction Eq. (1) exactly, and it allows a rigorous derivation of the α-model equation (2). So far only the nonhelical case has been extensively analyzed in the literature, e.g., in Refs. [10–12]. Here we address the problem in its full generality.

As an important new result, we show that the evolution equations for the magnetic energy and the magnetic helicity have self-adjoint structure (we note here that although the kinematic-dynamo equations have been known for over 20 years, their self-adjoint structure had so far not been discovered). As a consequence, in the kinematic regime, the spectrum of magnetic fluctuations can be expressed as a sum of eigenfunctions of a Schrödinger-type equation with imaginary time, where the eigenvalue λ gives the growth rate of the corresponding mode.

In analogy to the quantum-mechanical states in a potential well, the eigenmodes growing with $\lambda \leq 2\gamma_0$ correspond to "traveling particles;" i.e., they are correlated at the system size. By contrast, the faster-growing modes (with $\lambda > 2\gamma_0$) correspond to "trapped particles;" their correlation lengths are less than infinity, and they fill the whole range of scales down to the resistive scale. At any given scale the modes with $\lambda > 2\gamma_0$ may rapidly become dominant over the slowly growing nonlocalized modes.

The eigenmodes with $\lambda > 2\gamma_0$ are not captured by the mean-field equation (2); consequently the α-dynamo model (2), based on the assumption of scale separation and on small-scale smoothing, is inadequate. To describe the large-scale dynamo mechanism correctly, numerical simulations of uniform, isotropic, helical turbulence must resolve the full range of scales from l_0 to to l_η. Furthermore, the origin of large-scale fields, such as those observed in astrophysical situations, may be related to the nonzero large-scale average of the fluctuating part of the field, and not to the mean field as described by mean-field models.

The Kazantsev model for helical kinematic dynamo.— Kazantsev [10] and Kraichnan [13] introduced the solvable models in the theory of passive random advection. The essential assumption is that the random velocity field is Gaussian and short-time correlated. It is also assumed that the velocity field has zero mean, $\langle \mathbf{v} \rangle = 0$, so that the problem is completely specified by the velocity covariance tensor. For the statistically homogeneous and isotropic case, the covariance can be written as

$$\langle v^i(\mathbf{x}, t) v^j(\mathbf{x}', t') \rangle = \kappa^{ij}(|\mathbf{x} - \mathbf{x}'|)\delta(t - t'), \qquad (3)$$

where κ^{ij} is an isotropic tensor. For mirror-symmetric velocities, the correlation tensor κ^{ij} is symmetric with respect to the interchange of the indices i and j. In the general case, however, this tensor has both symmetric and antisymmetric parts,

$$\kappa^{ij}(x) = \kappa_N\left(\delta^{ij} - \frac{x^i x^j}{x^2}\right) + \kappa_L \frac{x^i x^j}{x^2} + g\epsilon^{ijk}x^k. \qquad (4)$$

The first two terms at the right-hand side of (4) represent the symmetric, nonhelical part, while the function $g(x)$ describes the helical part of the velocity correlation tensor. Here ϵ^{ijk} is the completely antisymmetric pseudotensor, and summation over the repeated indices is assumed. The requirement that the velocity be incompressible implies that $\kappa_N(x) = \kappa_L(x) + x\kappa_L'(x)/2$, where the primes denotes derivatives with respect to x.

The magnetic-field correlator can similarly be introduced: $H^{ij}(x, t) = \langle B^i(\mathbf{x}, t) B^j(0, t) \rangle$, satisfying

$$H^{ij} = M_N\left(\delta^{ij} - \frac{x^i x^j}{x^2}\right) + M_L \frac{x^i x^j}{x^2} + K\epsilon^{ijk}x^k, \qquad (5)$$

where the corresponding solenoidality constraint implies $M_N = M_L + xM_L'/2$. Our goal is to find the functions $M_L(x, t)$ and $K(x, t)$ that contain the information about the magnetic energy and the magnetic helicity.

Differentiating $H^{ij}(x, t)$ with respect to t and making use of (1), (3), and (4), we obtain after a cumbersome but straightforward calculation, that the magnetic correlation tensor obeys

$$\partial_t H^{ij} = \hat{R}^{imn}\hat{R}^{jrt}(T^{mr}H^{nt}), \qquad (6)$$

where $\hat{R}^{imn} = \epsilon^{ikl}\epsilon^{lmn}\nabla_k$, and $\nabla_k \equiv \partial/\partial x^k$. An analogous, although not identical, representation of this equation was derived in Ref. [11], but see also the derivation in Refs. [12,14]. The tensor T^{ij} can be represented in the following form:

$$T^{ij} = \frac{A}{\sqrt{2}}\left(\delta^{ij} - \frac{x^i x^j}{x^2}\right) + B\frac{x^i x^j}{x^2} + \frac{C}{\sqrt{2}}\epsilon^{ijk}\frac{x^k}{x}, \qquad (7)$$

where

$$A(x) = \sqrt{2}[\kappa_N(0) - \kappa_N(x) + 2\eta], \qquad (8)$$

$$B(x) = [\kappa_L(0) - \kappa_L(x) + 2\eta], \qquad (9)$$

$$C(x) = \sqrt{2}[g(0) - g(x)]x, \qquad (10)$$

and symbol $B(x)$ in (9) should not be confused with the magnetic field $B^i(x, t)$ in Eq. (5). Hereinafter, we adopt the notation $\kappa_0 \equiv \kappa_L(0) = \kappa_N(0)$, and $g_0 \equiv g(0)$.

Equation (6) can be considerably simplified, since the magnetic-field tensor (5) contains only two independent functions, M_L and K. The reduced equations were derived in Ref. [11], however, the symmetric structure of the tensor Eq. (6) was not preserved. In the next section, we derive the reduced equations, keeping their symmetric structure intact. In this way, we reveal the self-adjoint nature of the equations which allows us to gain new insight into the large-scale dynamo mechanism, and to elucidate the limitations of the conventional α-dynamo paradigm presented in the introduction.

The self-adjoint dynamo equations.—In this section we show that (6) is self-adjoint. We begin by rewriting (6) in the equivalent form,

$$\partial_t H^{ij} = \hat{D}^{ij}_{l\tilde{l}} J^{l\tilde{l}}_{nt} H^{nt}, \tag{11}$$

where $\hat{D}$ is the self-adjoint differential operator

$$\hat{D}\,^{ij}_{l\tilde{l}} = \epsilon^{ikl}\epsilon^{j\tilde{k}\tilde{l}}\nabla_k\nabla_{\tilde{k}}, \tag{12}$$

and the matrix J is symmetric,

$$J^{l\tilde{l}}_{nt} = \epsilon^{npl}\epsilon^{tq\tilde{l}}T^{pq}; \tag{13}$$

i.e., this matrix does not change under the interchange of its lower and upper sets of indices. We now express the operators $\hat{D}$ and J in the basis defined by the three orthogonal "vectors:"

$$\xi_1^{ij} = \frac{1}{x\sqrt{2}}\left(\delta^{ij} - \frac{x^i x^j}{x^2}\right), \tag{14}$$

$$\xi_2^{ij} = \frac{x^i x^j}{x^3}, \tag{15}$$

$$\xi_3^{ij} = \frac{1}{x\sqrt{2}}\epsilon^{ijs}\frac{x^s}{x}. \tag{16}$$

This is possible since the functions κ^{ij}, and H^{ij} can themselves be expanded in this basis. The normalization of the vectors $\xi_1^2 = \xi_2^2 = \xi_3^2 = 1/x^2$ is chosen in such a way as to preserve the self-adjoint structure of the differential operator, $\hat{D}$, as we will see presently. A straightforward calculation leads to

$$J = \begin{bmatrix} B & A & 0 \\ A & 0 & C \\ 0 & C & B \end{bmatrix}, \tag{17}$$

$$\hat{D} = \begin{bmatrix} \frac{\partial^2}{\partial x^2} & -\frac{\partial}{\partial x}\frac{\sqrt{2}}{x} & 0 \\ \frac{\sqrt{2}}{x}\frac{\partial}{\partial x} & -\frac{2}{x^2} & 0 \\ 0 & 0 & \frac{1}{x^2}\frac{\partial}{\partial x}x^4\frac{\partial}{\partial x}\frac{1}{x^2} \end{bmatrix}, \tag{18}$$

where both operators are manifestly self-adjoint.

We now make the following crucial observation. It can be verified directly that the operator $\hat{D}$ can be factorized as $\hat{D} = -\hat{R}\hat{R}^T$, where

$$\hat{R} = \begin{bmatrix} 0 & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\sqrt{2}}{x} & 0 \\ 0 & 0 & -\frac{1}{x^2}\frac{\partial}{\partial x}x^2 \end{bmatrix}. \tag{19}$$

This factorization immediately allows the dynamo equations to be transformed into self-adjoint form. Let us introduce the vector W such that $H = \hat{R}W$. As can be directly checked, with this definition the vector H automatically satisfies the solenoidality condition, i.e., its components in the basis (14)–(16) can be represented as $H = \{\sqrt{2}xM_N, xM_L, \sqrt{2}x^2K\}$, where $M_N = M_L + \frac{1}{2}xM_L'$, and M_L and K are some independent functions [compare this with (5)]. We further require that the vector W satisfy the equation

$$\partial_t W = -\hat{R}^T J\hat{R}W; \tag{20}$$

then, clearly, the function H obeys the dynamo equation (11), as can be verified by applying the operator $\hat{R}$ to both sides of Eq. (20). The operator in the right-hand side of (20) is now explicitly self-adjoint. This representation constitutes the formal solution of our problem.

For practical purposes, (20) can be further simplified since only two components of the vector H are independent. Conveniently, the necessary reduction is already present in (20). Indeed, calculating the operator in the right-hand side of (20), one sees that it acts only on the second and the third components of the vector W, so that the system is automatically reduced to the two independent equations that preserve the initial symmetry structure. The reduced equations have the self-adjoint form

$$\partial_t W = -\tilde{R}^T\tilde{J}\tilde{R}W, \tag{21}$$

where $\tilde{R}$ is the reduced form of the operator $\hat{R}$, and $\tilde{J}$ is the reduced form of J:

$$\tilde{R} = \begin{bmatrix} \frac{\sqrt{2}}{x} & 0 \\ 0 & -\frac{1}{x^2}\frac{\partial}{\partial x}x^2 \end{bmatrix}, \qquad \tilde{J} = \begin{bmatrix} \hat{E} & C \\ C & B \end{bmatrix}. \tag{22}$$

Here we introduced the self-adjoint operator

$$\hat{E} = -\frac{1}{2}x\frac{\partial}{\partial x}B\frac{\partial}{\partial x}x + \frac{1}{\sqrt{2}}(A - xA'). \tag{23}$$

The validity of (21) and (22) can be verified most easily by direct calculation of the right-hand sides of (20) and (21). For convenience, we write out the matrix form of Eq. (21) explicitly:

$$\begin{bmatrix} \partial_t W_2 \\ \partial_t W_3 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{x}\hat{E}\frac{\sqrt{2}}{x} & \frac{\sqrt{2}}{x^3}C\frac{\partial}{\partial x}x^2 \\ -x^2\frac{\partial}{\partial x}C\frac{\sqrt{2}}{x^3} & x^2\frac{\partial}{\partial x}\frac{B}{x^4}\frac{\partial}{\partial x}x^2 \end{bmatrix}\begin{bmatrix} W_2 \\ W_3 \end{bmatrix}, \tag{24}$$

and the relation $H = \tilde{R}W$ reads

$$M_L = \frac{\sqrt{2}}{x^2}W_2, \qquad K = -\frac{1}{\sqrt{2}x^4}\frac{\partial}{\partial x}(x^2W_3). \tag{25}$$

Equations (24) and (25) are the main result of this section.

For completeness, we note that the equations for the functions M_L and K were first derived by Vainshtein and Kichatinov [11] in the non-self-adjoint form:

$$\partial_t M = \frac{1}{x^4}\frac{\partial}{\partial x}\left(x^4\kappa\frac{\partial M}{\partial x}\right) + GM - 4hK, \tag{26}$$

$$\partial_t K = \frac{1}{x^4}\frac{\partial}{\partial x}\left(x^4\frac{\partial}{\partial x}(\kappa K + hM)\right). \tag{27}$$

Here we adopt the standard notation $\kappa = 2\eta + \kappa_L(0) - \kappa_L(x)$, $h = g(0) - g(x)$, $G = \kappa'' + 4\kappa'/x$, and $M \equiv M_L$. These equations also follow from (24). We also note that the Fourier-space versions of (26) and (27), were derived by Kulsrud and Anderson [15] for the limit of large magnetic Prandtl number (ratio of fluid viscosity to resistivity), and by Berger and Rosner [16] for the general case.

Discussion and conclusion.—In systems with no kinetic helicity [i.e., with $C(x) \equiv 0$], magnetic dynamo action is always possible if the magnetic Reynolds number is large enough [11,17]. Since the "helical" terms in (24) have a *destabilizing* effect, systems with kinetic helicity should exhibit dynamo action as well. A rigorous analysis of the dynamo mechanism requires knowledge of the exact spectrum of (24) which is not known for a general velocity correlator, $\kappa^{ij}(x)$. However, the typical behavior of the solution can be understood as follows. We introduce the mean-field growth rate, $\lambda_0 = g_0^2/\kappa_0$. From the asymptotic behavior of system (24) as $x \to \infty$, one can show that its eigenmodes with $\lambda > \lambda_0$ are localized, and the closer the growth rate to λ_0, the larger the correlation length. On the other hand, the eigenmodes corresponding to $\lambda \le \lambda_0$ have infinite correlation length.

The formal analogy between Eq. (24) and imaginary-time quantum mechanics suggests that the eigenfunctions with $\lambda > \lambda_0$ correspond to "particles" trapped by the potential provided by velocity fluctuations, while the eigenfunctions with $\lambda \le \lambda_0$ correspond to traveling particles. In the nonhelical case, where only trapped particles have positive eigenvalues, the spacing between the eigenvalues decreases with increasing magnetic Reynolds number; see, e.g., Ref. [12]. It is reasonable to expect that the same result holds for the helical case.

We now explain the extent to which the mean-field equation (2) describes the dynamo mechanism. Remarkably, in the Kazantsev model, Eq. (2) can be derived *exactly* (see, e.g., Refs. [4,14]), which allows one to find its precise relation to Eq. (24). In the derivation, $\bar{\mathbf{B}}(x, t)$ is the field averaged over the statistical ensemble of the velocity fluctuations (3), and the coefficients in the mean-field equation (2) are given by $\alpha = g_0$, $\beta = \eta + \kappa_0/2$ [14]. In this model, $\lambda_0 = 2\gamma_0$. Equation (2) can therefore be used to derive the evolution equation for the correlator of the mean field, $\bar{H}^{ij}(x, t) = \langle \bar{B}^i(x, t)\bar{B}^j(0, t)\rangle$, where the brackets denote averaging over the random initial conditions of the magnetic field.

It can be checked that this evolution equation formally coincides with the large-scale asymptotic ($x \to \infty$) of our system (24); consequently, it can only be used to obtain the large-scale asymptotics of the solutions of Eq. (24). More precisely, representing the magnetic field as $B^i(x, t) = \bar{B}^i(x, t) + \delta B^i(x, t)$, where δB^i is the fluctuating part, we can write $H^{ij}(x, t) = \langle \bar{B}^i(x, t)\bar{B}^j(0, t)\rangle + \langle \delta B^i(x, t)\delta B^j(0, t)\rangle$. We note that while the system (24) describes the exact function $H^{ij}(x, t)$, the mean-field equation (2) captures only its slowly growing nonfluctuating part $\bar{H}^{ij}$, which is described by the large-scale asymptotics of $H^{ij}(x, t)$, since the correlation of the fluctuations vanishes for infinite scale separation.

In summary, we have used the Kazantsev model to compare the exact spectra of magnetic energy and helicity with the predictions of the α model (2). We have demonstrated that the large-scale asymptotics ($x \to \infty$) of the exact solution is described by the nonlocalized eigenmodes ($\lambda \le \lambda_0$) of the self-adjoint dynamo Eq. (24). This asymptotics can also be derived from the mean-field α-dynamo equation (2). However, model (2) misses the faster-growing eigenmodes with $\lambda > \lambda_0$, which are present in (24). The correlation lengths of these eigenmodes are generally not small. They fill the range of scales from the system scale to the resistive ones, so these modes may not be removed by a small-scale smoothing procedure. In numerical simulations or astrophysical applications these modes may dominate the slowly growing "mean-field" modes. The correct description of the dynamo mechanism thus requires the resolution of the whole range of scales available to the magnetic field.

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