Stellar Stability by Thermodynamic Instability

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For gravity-dominated systems the three features shrinking ⇔ energy decrease ⇔ temperature increase are dynamically linked together. So are their inverses: expansion ⇔ energy increase ⇔ temperature decrease. We exhibit these features by one classical particle in a suitable environment, and by many particles with purely attractive interactions. We then show how the ensuing negative heat capacity tames an explosive energy input.

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Stars are threatened by two gigantic instabilities. The gravothermal effect (negative heat capacity) tries to implode the star, and thermonuclear reactions try to explode it. These two giants are shackled together and stabilize each other for billions of years. But beware, if eventually one of them gets weaker than the other. The following scenarios are known.

(1) The center of a star gets so dense that the electron gas becomes degenerate and the heat capacity becomes positive. Then, as soon as new nuclear material can burn, it does so in a huge nuclear explosion (helium flash). However, the outer part of the star still has a negative heat capacity and absorbs this energy by expansion and cooling. In this way the thermal instability extinguishes the nuclear fire, and the appearance of the star does not betray its serious digestion problems. We notice only a slight swelling of the belly.

(2) When the temperature reaches a level such that electrons may creep into the protons to create a neutron and a neutrino, the nuclear power feeds no surplus energy into the star. All excess energy is immediately carried away by the neutrinos, and electrons disappear to carry the weight of the star. Then nuclear power gives in and gravitation shows its might. The star collapses within seconds, and its luminosity exceeds the shine of its galaxy: A *supernova* appears, and a neutron star or a black hole is left over.

(3) The two giants age together, keeping equal strength. The star contracts to lose its negative heat capacity and exhausts its nuclear fuel. Then it turns peacefully into a white dwarf and, perhaps, into a Wigner crystal.

The phenomenon of negative specific heat of gravitational systems has a long history. The first hints can be found in the book *Gaskugeln* by Emden from 1907 [1]. In 1962 Antonov demonstrated that the phase-space volume below the energy shell diverges for $1/r$-potential systems with sufficiently many particles [2]. As a consequence, the heat capacity becomes negative. This fact was readily accepted in astronomical circles [3–5], but only recently sufficiently many examples have been found to establish it in physics [6,7]. There were some good reasons for this reluctance of physicists.

(I) In the canonical ensemble the specific heat is given by the square fluctuations of the energy and cannot become negative. The reason is simply [8] that as long as the specific heat is negative the system does not come into equilibrium with the heat reservoir. If the canonical and the microcanonical ensembles are equivalent, also the latter has positive specific heat. For Coulomb systems, this equivalence has been proved by Lieb and Lebowitz but still in some atomic spallation experiments traces of negative specific heat have been observed. This seeming contradiction may have the resolution that either the system is too far from the thermodynamic limit or it is not ergodic, and some ergodic components have negative specific heat [9].

(II) Thermodynamic stability means that the energy as a function of entropy, volume, and the number of particles is convex. This is guaranteed if, and only if, this function is subadditive and homogeneous of first degree. If all forces are attractive, subadditivity follows and thermodynamic stability becomes equivalent to homogeneity, which means that the energy is an extensive quantity [10]. It is the latter property that fails for gravitation-dominated systems.

First, we study the phenomenological energy evolution of a heated system and show how negative heat capacity turns a repellor into an attractor. Next, we show how a single particle in a constant gravitational field acquires a negative heat capacity by a clever jumping board. We can even realize the Bekenstein-Hawking thermodynamics [11,12]. Finally, we illustrate these effects by our favorite example for negative heat capacity, classical particles with attractive pair interactions. Once a cluster has been formed, we simulate the nuclear processes by heating the core of the cluster. In doing so, we leave the energy shell and go to higher energies. In stars this energy increase comes to an end, since it loses energy by radiation. We simulate this by cooling the particles that hit the container walls. In this way, the energy is decreased again and the two mechanisms together lead to the convergence of the energy to a final value. Plotting the temperatures gives the crazy picture that upon *heating* the temperature *decreases,* and only increases again once the *cooling* becomes effective. In this way, the two instabilities cancel each other and allow an equilibrium temperature to be reached.

Phenomenological thermodynamics.—To begin, we consider only the energy E and the temperature T of the system. Their changes are linked by the heat capacity c: $dE = cdT$. c is always considered to be a given constant.

Suppose, for simplicity, that we have an energy source $\sim nT$ and an energy sink $\sim -sT$ such that, in suitable time units, $dE/dt = (n-s)T$, and, thus, $dT/dt = (n-s)T/c$. This has as a solution $T(t) = T(0)e^{t(n-s)/c}$. Two observations follow immediately.

(I) For a normal system ($c > 0$) we get explosion, that is an exponential increase of the energy, if $n > s$; otherwise the system freezes.

(II) When the system has a negative heat capacity, $c < 0$, it is just the other way around. If n stands for the nuclear energy production, then this instability is tamed by the negative heat capacity of a star. On the other hand, if n ceases and only s (from neutrino production) remains, we get the supernova implosion.

Next we want to model the situation, where the nuclear energy production n sets in only above a critical temperature T_c. For simplicity, we represent the rate of energy emission by a constant s. We have to distinguish the cases.

(I) Positive heat capacity: In this case we take the energy production $n(E - E_c)$ for $E > E_c$, and 0 otherwise. The time evolution is given by $dE/dt = n(E - E_c) - s$ for $E > E_c$, and $dE/dt = -s$ if $E < E_c$. If $E(0) > E_c + s/n$, it is explosive, $E(t) = E(0)e^{nt} + (E_c + s/n)(1 - e^{nt})$. Otherwise, it decreases with time.

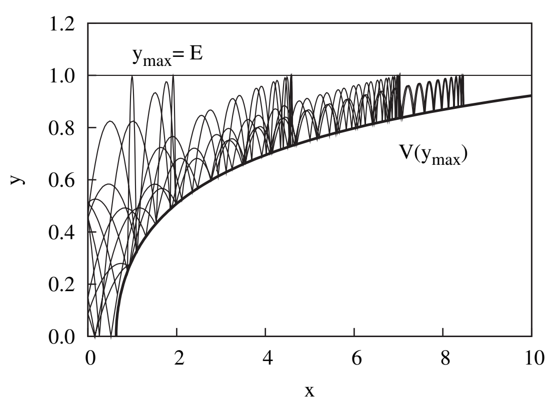

FIG. 1. Geometry and short trajectory for the jumping-board model. The gravitational force points into the negative y direction, and the particle is elastically reflected at the bottom, which is given by $V(y_{\max})$ as specified in the main text. Since the particle is also reflected at the line $x = 0$, the spatial domain may be restricted to positive x. The energy, $E = 1$ for this figure, corresponds to the maximum height, $y_{\max}$, the jumping particle may reach, if both momentum components p_x and p_y vanish.

(II) Negative heat capacity: Since the energy production is positive, we set it equal to $n(E_c - E)$ for $E < E_c$. The energy loss we keep to be s. Then, $dE/dt = n(E_c - E) - s$ for $E < E_c$, and $dE/dt = -s$ if $E > E_c$. Now the fixed point $E = E_c - s/n$ is approached exponentially: $E(t) = E(0) - (E_c - s/n)(1 - e^{-nt})$, if $E < E_c$; otherwise it decreases linearly with t. Thus, the negative heat capacity changes the bifurcation point $E_c + s/n$ of the unstable case I into the fixed point $E_c - s/n$ of the stable model II.

Two-dimensional billiard with gravitation.—Next, we construct a simple mechanical model, which consists of a single classical particle ($N = 1$) moving in the (x, y) plane, but confined to the region $-V(y) < x < V(y)$, $y > 0$, where $V > 0$ is to be specified later. In addition, there is a constant gravitational force in the y direction. In suitable units, the Hamiltonian is

$$H = p^2 + y, \tag{1}$$

which is positive in the allowed region. Figure 1 shows the geometry and a short trajectory of the particle, which is elastically reflected from the board V at the bottom. To avoid negative x, the particle is also reflected at $x = 0$. We demonstrate below that this model has a negative heat capacity. Of course, for a single particle many thermodynamic notions lose their meaning. Granting ergodicity, however, the statements about time averages in the microcanonical ensemble still hold. Our results are readily shown to carry over to an ensemble of ideal gas particles [13]. This may serve to justify our thermodynamic terminology also for $N = 1$.

If we denote the volume element in phase space, $dxdydp_xdp_y$, by $d\omega$, the volume of the energy shell is

$$\Omega(E) = \int d\omega\, \delta(H - E) = \pi \int_0^{E=y_{\max}} dy V(y), \tag{2}$$

where the p integral supplies just π. The entropy has to be defined by the volume underneath the energy shell in order to get the equipartition theorem:

$$\begin{aligned} e^{S(E)} &= \int_0^E dE' \Omega(E') = \int_0^E dy (E - y) V(y) \\ &= \int d\omega\, \delta(H - E) p^2, \end{aligned} \tag{3}$$

and, thus, the microcanonical average of the kinetic energy becomes the temperature:

$$\langle p^2 \rangle = \int d\omega\, \delta(H - E) p^2 / \Omega(E) = \frac{e^{S(E)}}{(de^{S(E)}/dE)} = T. \tag{4}$$

The following simple form of S leads to a negative heat capacity for $\gamma > 1$:

$$S = E^\gamma, \qquad T \equiv dE/dS = S^{-1+1/\gamma} 1/\gamma = E^{1-\gamma}/\gamma, \tag{5}$$

$$dT/dE = -(1 - 1/\gamma)E^{-\gamma} < 0. \quad (6)$$

$\gamma = 2$ corresponds to the Bekenstein-Hawking thermodynamics [11,12], but this form of S can be realized only in our model for big E, since it requires $e^{S(0)} = 0$. If V starts as E^α, $\alpha > 0$, we have to add a term proportional to $\ln E$:

$$S = E^\gamma + (\alpha + 2)\ln E.$$

This leads to $S' = \gamma E^{\gamma-1} + (\alpha + 2)E^{-1}$ and $S'' = \gamma(\gamma - 1)E^{\gamma-2} - (\alpha + 2)E^{-2}$. Thus, for sufficiently large E we still get $S'' > 0$ and, thus, a negative heat capacity. The form of the jumping board is determined by Eq. (3),

$$V(E) = d^2/dE^2(e^{S(E)}) = d^2/dE^2(E^{\alpha+2}\exp(E^\gamma))/\pi,$$

which is positive as required. For $\gamma = 2$ and $\alpha = 0$, the case closest to the black hole thermodynamics, we get

$$V(E) = (2 + 10E^2 + 4E^4)\exp(E^2)/\pi.$$

Since $E = y_{\max}$, V may be viewed as a function of $y_{\max}$ as is shown in Fig. 1. Furthermore,

$$T = E(2 + 2E^2)^{-1},$$
$$c = dE/dT = (2 - 2E^2)^{-1}(2 + 2E^2)^2 < 0 \quad \text{for } E > 1.$$

In Fig. 2 we compare the theoretical dependence of the temperature on the total energy (smooth line) with computer simulation results (points). The regime of negative specific heat, $E > 1$, is clearly visible. The fluctuations for large energies are due to the difficulty to achieve ergodicity. The intuition behind this example of negative heat capacity was given in Ref. [14]. An increase of E opens so much available space in the x direction, such that the volume of the energy shell increases more than exponentially with E. This makes $S''(E) > 0$, which in turn implies $c < 0$.

Purely attracting particles.—Finally, we consider a two-dimensional system of $N = 400$ classical particles in an elastically reflecting circle (radius R), which interact with a short-ranged negative Gauss potential [15], $v(r) = -\epsilon\exp\{-r^2/\sigma^2\}$. Here, $r = |x_i - x_j|$ is the separation between particles i and j, and $x_i \in R^2$. For our numerical work reduced units are used, for which the particle mass m, the potential parameters ϵ and σ, and Boltzmann's constant k are unity.

The simulations are carried out in three steps.

First (equilibration) step: A homogeneous gas with a particle density $\rho \equiv N/\pi(R/\sigma)^2 = 0.5$, and with random velocities is set up as the initial condition. The initial temperature, $T_0 \equiv \sum_{i=1}^N m_i v_i^2/k$, is taken to be one. This unstable state is propagated forward in time at a constant energy E up to a time $t_1 = 3000$, at which the system is close to equilibrium and consists of a single large cluster of particles floating in a gas of the remaining particles. See Fig. 3.

Two particles i and j are considered to be members of the same cluster, if they are separated by less than $\sigma/\sqrt{2}$. For the temperature of the largest cluster we take $T_c = \sum' m(v_i - v_c)^2/k$, where the sum is over all particles belonging to the largest cluster, and where v_c is its center-of-mass velocity. The history of T_c during the equilibration step is shown by curve A in Fig. 4. One observes a steep initial increase due to the formation of the main cluster, until a stationary state is approached, which persists for times $t > t_1$ as shown. The corresponding total energy, E, is constant (line A in Fig. 5).

Second (heating) step: For times $t_1 < t < t_2$, $t_2 = 4800$, the center of the cluster is heated by adding a friction term, $-\zeta w(|x_i - x_c|)p_i$, to the equations of motion for particle i,

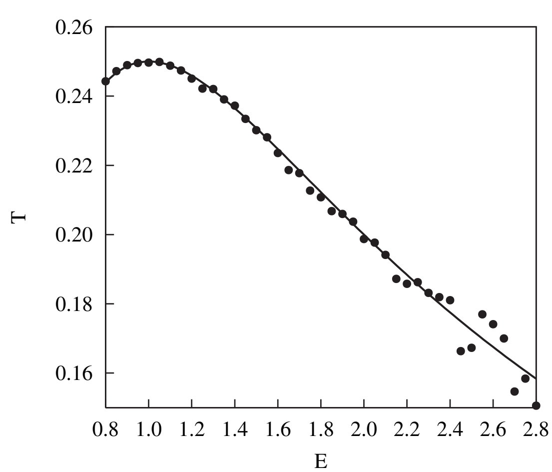

FIG. 2. Dependence of the temperature on the total energy for the mechanical jumping-board model. Simulation results (points) are compared to the theoretical expression (solid line). The number of computed jumps is gradually increased from 4×10^6, for $E = 0.8$, to 40×10^6, for $E = 2.8$.

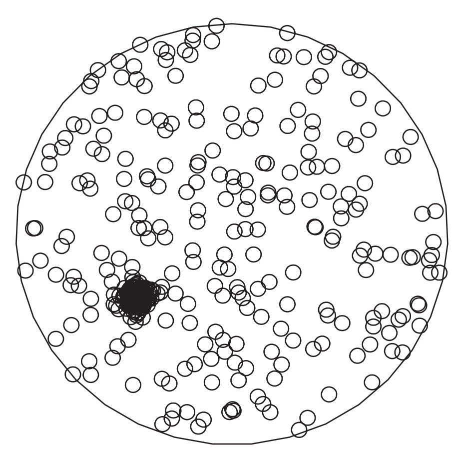

FIG. 3. Negative Gauss potential: typical particle configuration at a time $t_0 = 3000$. The particles are represented by circles with a diameter equal to unity.

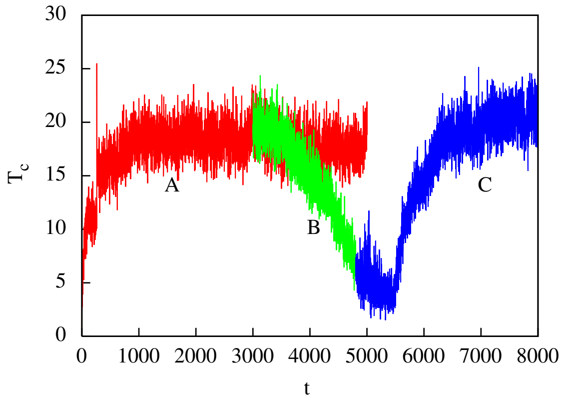

FIG. 4 (color online). Time evolution for the temperature T_c of the largest cluster. Line A belongs to the equilibration step, line B to the heating step, and line C to the final heating-cooling step.

$$\dot{x}_i = v_i, \qquad \dot{v}_i = F_i/m - \zeta w(|x_i - x_c|)v_i,$$

with a *negative* friction constant $\zeta = -0.001$. Here, F_i is the total force on i, and v_i is its velocity. The normalized weight function w is centered on the center of mass of the main cluster, x_c, and has a cutoff radius $h = 1$ [16],

$$w(r) = \begin{cases} \frac{5}{\pi h^2}(1 + 3\frac{r}{h})(1 - \frac{r}{h})^3 & \text{for } r < h \\ 0 & \text{for } r \geq h, \end{cases}$$

As a consequence of heating the cluster core, the energy of the system is increased (curve B in Fig. 5). At the same time, line B in Fig. 4 indicates that the temperature of the main cluster decreases. This is a spectacular consequence of the negative heat capacity of the system.

In the absence of any energy loss, the energy of the system goes up until a transition energy is reached, at which the cluster dissolves. For still higher energies, the system behaves as an ideal gas with a positive specific heat, for which the temperature increases with the energy. In our simulation we avoid this phase transition by adding a loss mechanism at time t_2.

Third (heating+cooling) step: For times $t > t_2 = 4800$ and in addition to the heating, the system is also subjected to a cooling process at the boundary, which is supposed to mimic the radiation loss off a star. Whenever a particle is reflected at the circular boundary, both of its velocity components are reduced by 0.5%. As a consequence, the total energy of the system drops again (line C in Fig. 5). After some delay due to the weak gas particle–cluster interactions, also the temperature of the main cluster responds and goes up again (line C in Fig. 4). The cluster temperature eventually stabilizes at a very high level. This nicely demonstrates how energy loss is responsible for raising the cluster temperature again, pushing the unstable matter towards a stationary nonequilibrium state with a high cluster temperature T_c.

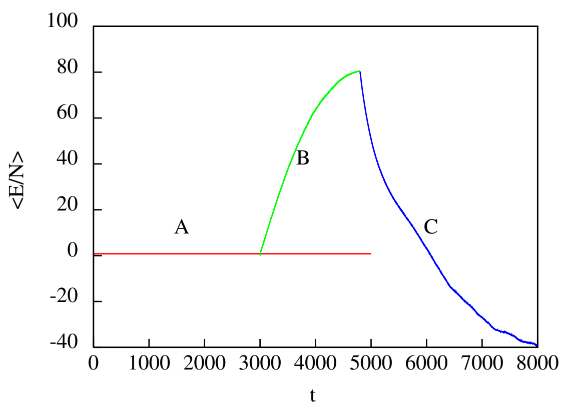

FIG. 5 (color online). Time evolution for the energy E of the system. The lines A, B, and C refer to the respective equilibration, heating, and heating-cooling steps mentioned in the main text.

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