Confinement and Superfluidity in One-Dimensional Degenerate Fermionic Cold Atoms

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The physical properties of arbitrary half-integer spins $F = N - 1/2$ fermionic cold atoms trapped in a one-dimensional optical lattice are investigated by means of a low-energy approach. Two different superfluid phases are found for $F \geq 3/2$ depending on whether a discrete symmetry is spontaneously broken or not: an unconfined BCS pairing phase and a confined molecular-superfluid instability made of $2N$ fermions. We propose an experimental distinction between these phases for a gas trapped in an annular geometry. The confined-unconfined transition is shown to belong to the $\mathbb{Z}_N$ generalized Ising universality class. We discuss the possible Mott phases at $1/2N$ filling.

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In the recent past, spectacular experimental progress has allowed us to cool down alkali atoms below Fermi temperatures offering the promising perspective to study strongly correlated electronic effects, such as high-temperature superconductivity, in a new context [1]. On top of strong correlations, ultracold atomic systems provide also an opportunity to investigate the effect of spin degeneracy since in general alkali fermionic atoms have (hyperfine) spins $F > 1/2$ in their lowest hyperfine manifold. The simplest examples are spin-3/2 atoms such as ^{9}Be, ^{132}Cs, and ^{135}Ba atoms. In this Letter, we elucidate the generic physical features of half-integer spins $F = N - 1/2$ fermionic cold atoms in the particular case of a one-dimensional optical lattice. The low-energy physical properties of these systems are known to be described by a Hubbard-like Hamiltonian [2]:

$$\mathcal{H} = -t\sum_{i,\alpha}[c^\dagger_{\alpha,i}c_{\alpha,i+1} + \text{H.c.}] + \sum_{i,J} U_J \sum_{M=-J}^{J} P^\dagger_{JM,i}P_{JM,i}, \tag{1}$$

where $c^\dagger_{\alpha,i}$ ($\alpha = 1, \ldots, 2N$) is the fermion creation operator corresponding to the $2F + 1 = 2N$ atomic states. The pairing operators in Eq. (1) are defined through the Clebsch-Gordan coefficient for two indistinguishable particles: $P^\dagger_{JM,i} = \sum_{\alpha\beta}\langle JM|F, F; \alpha, \beta\rangle c^\dagger_{\alpha,i}c^\dagger_{\beta,i}$. The interactions are SU(2) spin conserving and depend on U_J parameters corresponding to the total spin of two spin-F particles which takes only even integers value due to Pauli's principle: $J = 0, 2, \ldots, 2N - 2$. Even in this simple scheme the interaction pattern is still involved. It is thus highly desirable to focus on a much simpler paradigmatic model that incorporates the relevant physics of higher-spin degeneracy. In this respect, we consider a two coupling-constant version of model (1) with $U_2 = \cdots = U_{2N-2} \neq U_0$:

$$\mathcal{H} = -t\sum_{i,\alpha}[c^\dagger_{\alpha,i}c_{\alpha,i+1} + \text{H.c.}] + \frac{U}{2}\sum_i \rho_i^2 + V\sum_i \pi_i^\dagger \pi_i, \tag{2}$$

$\rho_i = \sum_\alpha c^\dagger_{\alpha,i}c_{\alpha,i}$ being the particle number operator and $U = 2U_2$, $V = U_0 - U_2$. In Eq. (2), the singlet BCS pairing operator for spin-F fermions is $\pi_i^\dagger = \sqrt{2N}P^\dagger_{00,i} = c^\dagger_{\alpha,i}\mathcal{J}_{\alpha\beta}c^\dagger_{\beta,i}$, the matrix $\mathcal{J}$ being the natural generalization of the familiar antisymmetric tensor $\epsilon = i\sigma_2$ to spin $F > 1/2$. Such a singlet-pairing operator has been extensively studied in the context of two-dimensional frustrated quantum magnets [3]. When $V = 0$, i.e., $U_0 = U_2$, model (2) corresponds to the SU($2N$) Hubbard model. The Hamiltonian (2) for $V \neq 0$ still displays a large symmetry since it is invariant under the Sp($2N$) group which consists of unitary matrices U that satisfy $U^* \mathcal{J} U^\dagger = \mathcal{J}$ [4]. In the spin $F = 1/2$ case, model (2) reduces to the SU(2) Hubbard chain since SU(2) $\simeq$ Sp(2). In the spin-3/2 case, models (1) and (2) have an exact SO(5) $\simeq$ Sp(4) symmetry [4]. This Sp($2N$) symmetry considerably simplifies the problem but may appear rather artificial. However, we expect that, for generic and small interactions, the original SU(2) spin-rotational invariance will be dynamically enlarged at sufficiently low energy [5,6]. A second reason to consider the Sp($2N$) symmetric model (2) stems from the fact that the Sp($2N$) and SU(2) groups share the same center, the $\mathbb{Z}_2$ group. Moreover, the striking physical properties of the system rely on the existence of a $\mathbb{Z}_N$ symmetry which is *also* a symmetry of the SU(2) model (1). This $\mathbb{Z}_N$ symmetry, which simply amounts to a global redefinition of the fermion phase, is properly defined as the coset between the center $\mathbb{Z}_{2N}$ of the SU($2N$) group and the center $\mathbb{Z}_2$ of the Sp($2N$) or SU(2) one: $\mathbb{Z}_N = \mathbb{Z}_{2N}/\mathbb{Z}_2$ with

$$\mathbb{Z}_{2N}: c_{\alpha,i} \rightarrow e^{\mathrm{i}n\pi/N} c_{\alpha,i}, \qquad n = 0, \ldots, 2N-1, \tag{3}$$

the $\mathbb{Z}_2$ symmetry being $c_{\alpha,i} \rightarrow -c_{\alpha,i}$. The $\mathbb{Z}_N$ symmetry, defined by Eq. (3) with $n = 0, \ldots, N-1$, provides an important physical ingredient not present in the $F = 1/2$ case. The stabilization of a quasi-long-range BCS phase for $F > 1/2$ requires the spontaneous breaking of this $\mathbb{Z}_N$ symmetry since the singlet pairing $\pi_i^\dagger$ is *not* invariant under this symmetry. Since $\mathbb{Z}_N$ may also be viewed as a discrete subgroup of the global U(1) charge symmetry [see Eq. (3)], it is tempting to interpret, for $F > 1/2$, the breaking of $\mathbb{Z}_N$ as the reminiscence of the spontaneous global U(1) charge breaking that characterizes the BCS phase in higher dimensions. In contrast, if $\mathbb{Z}_N$ is not broken, the BCS instability is suppressed and the leading superfluid instability, which has to be a singlet under the $\mathbb{Z}_N$ symmetry, is a molecular object made of $2N$ fermions. In the following, the delicate competition between these superfluid instabilities will be investigated by means of a low-energy approach.

Phase diagram.—The low-energy effective field theory associated with Eq. (2) is obtained, as usual, from the continuum description of the lattice electronic operators in terms of right and left moving Dirac fermions: $c_{\alpha,i}/\sqrt{a_0} \rightarrow R_\alpha(x)e^{ik_F x} + L_\alpha(x)e^{-ik_F x}$, $x = ia_0$, a_0 being the lattice spacing, and k_F is the Fermi momentum [7]. Away from half filling (i.e., N atoms per site), it separates into two commuting density and spin pieces: $\mathcal{H} = \mathcal{H}_d + \mathcal{H}_s$ with $[\mathcal{H}_d, \mathcal{H}_s] = 0$. The U(1) density sector is described by a bosonic field Φ and its dual Θ whose dynamics is governed by the free-boson Hamiltonian:

$$\mathcal{H}_d = \frac{v}{2}\left[\frac{1}{K_d}(\partial_x \Phi)^2 + K_d(\partial_x \Theta)^2\right], \tag{4}$$

where $v = v_F\{1 + [2V + UN(2N-1)]/(N\pi v_F)\}^{1/2}$ [$v_F = 2ta_0 \sin(k_F a_0)$ being the Fermi velocity] and $K_d = \{1 + [2V + UN(2N-1)]/(N\pi v_F)\}^{-1/2}$ are the Luttinger parameters. The conserved quantities in this U(1) sector are the total particle number and current: $\mathcal{N} = \int dx (R_\alpha^\dagger R_\alpha + L_\alpha^\dagger L_\alpha) = \sqrt{2N/\pi}\int dx \partial_x \Phi$ and $\mathcal{J} = \int dx(-R_\alpha^\dagger R_\alpha + L_\alpha^\dagger L_\alpha) = \sqrt{2N/\pi}\int dx\partial_x \Theta$, respectively. For incommensurate fillings, the density degrees of freedom are massless and display metallic properties in the Luttinger liquid universality class [7]. All nontrivial physics is encoded in the spin part of the Hamiltonian:

$$\begin{aligned}\mathcal{H}_s = \frac{2\pi v_s}{2N+1}&[I^a_{\parallel R} I^a_{\parallel R} + I^i_{\perp R} I^i_{\perp R} + R \rightarrow L] \\ &+ g_\parallel I^a_{\parallel R} I^a_{\parallel L} + g_\perp I^i_{\perp R} I^i_{\perp L},\end{aligned} \tag{5}$$

where $g_\parallel = -2(2V + NU)/N$, $g_\perp = 2(2V - NU)/N$, and we have neglected a velocity anisotropy. The Hamiltonian (5) describes a SU($2N$)$_1$ conformal field theory (CFT) perturbed by a marginal current-current interaction. In Eq. (5), the currents $I^A_{R(L)}$, $A = (1, \ldots, 4N^2 - 1)$, of the SU($2N$)$_1$ CFT have been decomposed into $\parallel$ and $\perp$ parts $I^A = (I^a_\parallel, I^i_\perp)$ with respect to the Sp($2N$) symmetry of the lattice model (2). The currents $I^a_{\parallel R(L)}$, $a = 1, \ldots, N(2N+1)$ generate the Sp($2N$)$_1$ CFT symmetry and can be simply expressed in terms of the chiral Dirac fermions: $I^a_{\parallel R} = R^\dagger_\alpha T^a_{\alpha\beta} R_\beta$, T^a being the generators of Sp($2N$) in the fundamental representation. The remaining SU($2N$)$_1$ currents are written as $I^i_{\perp R} = R^\dagger_\alpha T^i_{\alpha\beta} R_\beta$, $i = 1, \ldots, 2N^2 - N - 1$ (similar expressions hold for the left currents). The next step of the approach is to consider a description which singles out the $\mathbb{Z}_N$ symmetry of the lattice model (2) discussed above. To this end, we shall use the quantum equivalence [8]: U($2N$)$_1 \rightarrow$ U(1) $\times$ Sp($2N$)$_1 \times \mathbb{Z}_N$, $\mathbb{Z}_N$ being the parafermion CFT which describes self-dual critical points of two-dimensional $\mathbb{Z}_N$ Ising models [9]. This conformal embedding provides us with a *nonperturbative* basis to express any physical operator in terms of its density and spin degrees of freedom which are described, respectively, by the U(1) and Sp($2N$)$_1 \times \mathbb{Z}_N$ CFTs. The lattice $\mathbb{Z}_N$ symmetry is then captured, within this low-energy approach, by an effective 2D $\mathbb{Z}_N$ model which is a generalization to $N > 2$ of the standard Ising model. As in the $N = 2$ case, these $\mathbb{Z}_N$ Ising models exhibit two phases described by order and disorder parameters σ_k and μ_k, $k = 1, \ldots, N-1$, which are dual to each other by means of the Kramers-Wannier (KW) duality symmetry. This duality transformation maps the $\mathbb{Z}_N$ symmetry, which is broken in the low-temperature phase ($\langle\sigma_k\rangle \neq 0$ and $\langle\mu_k\rangle = 0$), onto a $\tilde{\mathbb{Z}}_N$ symmetry, which is broken in the high-temperature phase where $\langle\mu_k\rangle \neq 0$ and $\langle\sigma_k\rangle = 0$. At the critical point, the theory is self-dual with a $\mathbb{Z}_N \times \tilde{\mathbb{Z}}_N$ symmetry and its universal properties are captured by the $\mathbb{Z}_N$ parafermion CFT [9]. In the simplest $N = 2$ case, there is a simple free-field representation of the unperturbed SU(4)$_1$ CFT in terms of real fermions, which has been extensively used in the context of two-leg ladders [7]. Introducing real fermions $\xi^0_{R,L}$ and $\xi^a_{R,L}$, $a = 1, \ldots, 5$ to describe, respectively, the $\mathbb{Z}_2$ and SO(5)$_1 \simeq$ Sp(4)$_1$ CFTs, the interacting part of Eq. (5) becomes $\mathcal{H}^{\rm int}_{\rm s} = g_\parallel(\xi^a_R \xi^a_L)^2 + g_\perp \xi^0_R \xi^0_L \xi^a_R \xi^a_L$. The latter model has been studied recently to describe a SO(5) symmetric two-leg ladder [10]. For generic N, the phase diagram of Eq. (5) can be elucidated by means of a two-loop renormalization group (RG) analysis. As depicted in Fig. 1 it consists of three regions [11]. Region I is a generalized spin-density-wave (SDW) phase which is obtained when U and V are positive. In that case, $g_{\perp,\parallel} \rightarrow 0$ in the infrared (IR) limit and the interaction is marginal irrelevant. Up to a spin-velocity anisotropy, the low-energy properties of this phase are similar to that of the repulsive SU($2N$) Hubbard chain with $2N - 1$ gapless spin excitations [12,13]. In contrast, a spin gap opens in the two remaining regions which are distinguished by the $\mathbb{Z}_N$ symmetry. In phase II, defined by $U < 0$ and $V > NU/2$, the RG flow in the far IR limit is

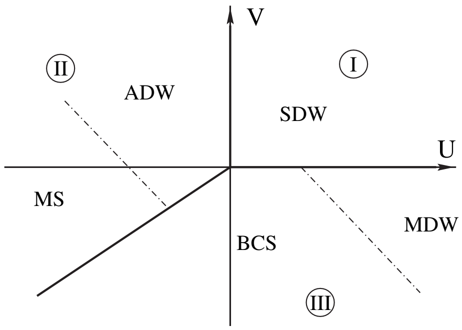

FIG. 1. Phase diagram of model (2). Dashed lines denote crossover lines, whereas the solid line marks the phase transition in the $\mathbb{Z}_N$ universality class between phase II and phase III.

attracted along a special symmetric ray $g_\parallel = g_\perp = g^* > 0$ where the interacting part of the Hamiltonian (5) can be rewritten in a SU(2N) invariant form:

$$\mathcal{H}_{\mathrm{s}}^{\mathrm{int}} = g^*(I^a_{\parallel R} I^a_{\parallel L} + I^i_{\perp R} I^i_{\perp L}) = g^* I^A_R I^A_L. \qquad (6)$$

The Hamiltonian (6), which governs the IR properties of phase II, takes the form of the SU(2N) Gross-Neveu (GN) model which is an integrable massive field theory [14]. It is instructive to have a simpler understanding of the spin-gap formation from the underlying $\mathbb{Z}_N$ Ising model. The $\mathbb{Z}_N$ and $\tilde{\mathbb{Z}}_N$ symmetries that define the low-T and high-T phases of this Ising model admit a representation in terms of the fermions $(R, L)_\alpha$. Indeed, the $\mathbb{Z}_N$ group (3) can be generated in this chiral fermion basis with help of the unitary operator $\mathcal{U} = e^{i\pi \mathcal{N}/N}$: $\mathcal{U} R(L)_\alpha \mathcal{U}^\dagger = e^{i\pi/N} R(L)_\alpha$. Similarly, the dual $\tilde{\mathbb{Z}}_N$ symmetry can be defined by $\tilde{\mathcal{U}} = e^{i\pi J/N}$: $\tilde{\mathcal{U}} R(L)_\alpha \tilde{\mathcal{U}}^\dagger = e^{\mp i\pi/N} R(L)_\alpha$. The ground state of model (6) displays long-range order associated with the order parameter $\mathrm{Tr}(g) = R^\dagger_\alpha L_\alpha e^{i\sqrt{2\pi/N}\Phi}$. In phase II, we find that the $\mathbb{Z}_{2N} = \mathbb{Z}_2 \times \mathbb{Z}_N$ symmetry remains unbroken while $\tilde{\mathbb{Z}}_N$ is spontaneously broken. The $\mathbb{Z}_N$ Ising model thus belongs to its high-T phase and a spectral gap is formed. In the second spin-gapped phase (III) of Fig. 1, defined by $V < 0$ and $V < NU/2$, the RG flow is now attracted along the asymptote: $g_\parallel = -g_\perp = g^* > 0$. In that case, the interacting part of the IR Hamiltonian becomes

$$\mathcal{H}_{\mathrm{s}}^{\mathrm{int}} = g^*(I^a_{\parallel R} I^a_{\parallel L} - I^i_{\perp R} I^i_{\perp L}), \qquad (7)$$

which can be recast as a SU(2N) GN model (6) by means of a *duality* transformation $\mathcal{D}$ on the fermions: $\mathcal{D} R(L) \mathcal{D}^{-1} = \tilde{R}(\tilde{L})$ with $\tilde{R}_\alpha = J_{\alpha\beta} R^\dagger_\beta$ and $\tilde{L}_\alpha = L_\alpha$. This transformation acts on the currents as $\tilde{I}^a_{\parallel R(L)} = I^a_{\parallel R(L)}$ and $\tilde{I}^i_{\perp R(L)} = -(+) I^i_{\perp R(L)}$ so that $\mathcal{D}$ indeed maps (7) onto (6). Besides the opening of a spectral gap, we thus find that phase III possesses a hidden symmetry at low energy, i.e., a $\widetilde{\mathrm{SU}}(2N)$ symmetry generated by the dual currents $(\tilde{I}^a_{\parallel R(L)}, \tilde{I}^i_{\perp R(L)})$. In fact, one has $\mathcal{D} \mathcal{U} \mathcal{D}^{-1} = \tilde{\mathcal{U}}$ so that $\mathcal{D}$ identifies to the KW duality of the $\mathbb{Z}_N$ Ising model. In phase III, the latter model thus belongs to its low-T phase and the $\mathbb{Z}_N$ symmetry is spontaneously broken whereas $\tilde{\mathbb{Z}}_{2N} = \mathbb{Z}_2 \times \tilde{\mathbb{Z}}_N$ remains unbroken. In summary, the existence of these two distinct spin-gapped phases is a nontrivial consequence of higher-spin degeneracy and does not occur in the $F = 1/2$ case. The emergence of the spin-gap stems from the spontaneous breakdown of the $\mathbb{Z}_N$ or $\tilde{\mathbb{Z}}_N$ discrete symmetries. As we shall see now, these symmetries are central to the striking physical properties displayed by these phases.

Spin superfluidity.—The low-energy properties of the spin sector of phase II can be extracted from the integrability of the SU(2N) GN model (6). Its spectrum consists of $2N - 1$ branches that transform in the SU(2N) representations [14]. These eigenstates are labeled by quantum numbers associated with the conserved quantities of the SU(2N) low-energy symmetry: $Q^a_\parallel = \int dx (I^a_{\parallel L} + I^a_{\parallel R})$, $a = (1, \ldots, N)$, and $Q^i_\perp = \int dx (I^i_{\perp L} + I^i_{\perp R})$, $i = (1, \ldots, N-1)$. Because of the Sp(2N) symmetry of model (2), the $Q^a_\parallel$ numbers are conserved whereas the $Q^i_\perp$ charges are only good quantum numbers at low energy. The spin spectrum in phase III can be obtained from the duality symmetry $\mathcal{D}$ and consists of $2N - 1$ branches which transform in the representations of the dual group $\widetilde{\mathrm{SU}}(2N)$. The dual quantum numbers are now given by $\tilde{Q}^a_\parallel = Q^a_\parallel$ and $\tilde{Q}^i_\perp = \int dx (\tilde{I}^i_{\perp L} + \tilde{I}^i_{\perp R}) = \int dx (I^i_{\perp L} - I^i_{\perp R}) = J^i_\perp$. We thus observe that the low-lying excitations in phase III carry quantized spin *currents* in the "$\perp$" direction. In this sense, phase III might be viewed as a partially spin-superfluid phase.

Confinement.—We shall now determine the nature of the dominant electronic instabilities of the different phases of Fig. 1. To this end let us consider operators $\mathcal{O}_{n,j}$ which carry particle number n and current j: $[\mathcal{N}(\mathcal{J}), \mathcal{O}_{n,j}] = n(j)\mathcal{O}_{n,j}$. Using the $\mathbb{Z}_{2N}$ and $\tilde{\mathbb{Z}}_{2N}$ generators, $\mathcal{U}$ and $\tilde{\mathcal{U}}$, we find that $\mathcal{O}_{n,j}$ carry $\mathbb{Z}_{2N}$ and $\tilde{\mathbb{Z}}_{2N}$ charges n and j, respectively. In phase II, the full $\mathbb{Z}_{2N} = \mathbb{Z}_2 \times \mathbb{Z}_N$ symmetry (3) is unbroken so that it costs a finite energy gap to excite states that either break the $\mathbb{Z}_2$ or $\mathbb{Z}_N$ symmetries. The dominant instabilities must thus be neutral under the $\mathbb{Z}_{2N}$ symmetry and the resulting order parameters $\mathcal{O}_{n,j}$ are characterized by $n = 0 \bmod 2N$ and $j = 0 \bmod 2$. In particular, there is no quasi-long-range BCS order in phase II since the lattice singlet-pairing operator $\pi^\dagger_i$ carries a charge $n = 2$ under the $\mathbb{Z}_{2N}$ symmetry (3). The $\mathbb{Z}_{2N}$ symmetry thus confines the electronic charge to multiples of $2Ne$, i.e., the leading superfluid instability in phase II is a composite object made of $2N$ fermions. In this respect, the dominant order parameters in phase II are $\rho_{2k_F} = L^\dagger_\alpha R_\alpha$ and $\Pi^{2N\dagger}_0 = \epsilon^{\alpha_1 \cdots \beta_N} R^\dagger_{\alpha_1} \cdots R^\dagger_{\alpha_N} L^\dagger_{\beta_1} \cdots L^\dagger_{\beta_N}$, which are, respectively, the $2k_F$ component of the atomic density ρ and the uniform component of the lattice SU(2N)-singlet superconducting instability made of $2N$ fermions: $\Pi^{2N\dagger}(i) = \epsilon^{\alpha_1 \cdots \alpha_{2N}} c^\dagger_{\alpha_1, i} \cdots c^\dagger_{\alpha_{2N}, i}$. These orders are power-law fluctuating: $\langle \rho^\dagger_{2k_F}(x)\rho_{2k_F}(0)\rangle \sim x^{-K_d/N}$, and $\langle \Pi_0^{2N\dagger}(x)\Pi_0^{2N}(0)\rangle \sim x^{-N/K_d}$. For $K_d < N$, the leading instability is ρ_{2k_F}, which gives rise to an atomic-density-wave (ADW) phase, whereas for $K_d > N$ a SU($2N$) molecular-superfluid (MS) phase is stabilized (see Fig. 1) with order parameter $\Pi_0^{2N\dagger}$. The properties of phase III are obtained from those of phase II with the help of the duality symmetry $\mathcal{D}$: ($\mathcal{N} \leftrightarrow \mathcal{J}$, $\mathbb{Z}_N \leftrightarrow \tilde{\mathbb{Z}}_N$, $K_d \leftrightarrow 1/K_d$). Low-energy excitations in phase III carry now $n = 0 \bmod 2$ and $j = 0 \bmod 2N$ since the symmetry $\tilde{\mathbb{Z}}_{2N} = \mathbb{Z}_2 \times \tilde{\mathbb{Z}}_N$ remains unbroken. We find now the confinement of atomic currents and the emergence of a quasi-long-range BCS pairing phase. Under the duality $\mathcal{D}$ symmetry, the ADW phase is mapped onto a BCS phase for $K_d > 1/N$ with order parameter $\pi_0^\dagger = R^\dagger_\alpha \mathcal{J}_{\alpha\beta} L^\dagger_\beta$, whereas the MS phase is mapped onto a molecular density-wave (MDW) phase with order parameter $\bar{\rho}^{2N}_{2Nk_F} = \epsilon^{\alpha_1\cdots\beta_N}\mathcal{J}_{\alpha_1\gamma_1}\cdots\mathcal{J}_{\alpha_N\gamma_N}R_{\gamma_1}\cdots R_{\gamma_N}L^\dagger_{\beta_1}\cdots L^\dagger_{\beta_N}$ which emerges when $K_d < 1/N$. The spontaneous breaking of the $\mathbb{Z}_N$ symmetry (3) thus accounts for the emergence of the BCS superfluid phase and the spin-superfluidity phenomenon discussed above. The possible occurrence of two different superfluid phases II and III may be probed experimentally. Consider, for example, a gas trapped in an optical potential of length L, with an *annular* geometry and moving with tangential velocity V. This amounts to imposing a total particle current in the system $J = 4NV/V_0$, where $V_0 = h/mL$. In the superfluid phase III, since the low-energy excitations carry currents $j = 0 \bmod 2N$, we expect the total energy $E(V)$ to display degenerate minima for quantized velocities: $V_n = nV_0/2$ irrespective of the value of the spin F. In contrast, in the phase II where currents are unconfined, we expect the degenerate minima of $E(V)$ at $V_n = nV_0/2N$.

The $\mathbb{Z}_N$ phase transition.—The nature of the quantum phase transition between the two spin-gapped phases II and III can be determined through the duality symmetry $\mathcal{D}$. On the self-dual line $g_\perp = 0$, i.e., $2V = NU$, there is a separation of the Sp($2N$) and $\mathbb{Z}_N$ degrees of freedom. Though the Sp($2N$) sector remains gapful when $U < 0$, the effective $\mathbb{Z}_N$ Ising model is at its self-dual critical point and governs the phase transition. The $\mathbb{Z}_N$ quantum criticality for $N = 2, 3$ may be revealed by considering the ratio $\mathcal{R}_N(x) = (\langle \pi_0^\dagger(x)\pi_0(0)\rangle)^{N^2}/\langle \Pi_0^{2N\dagger}(x)\Pi_0^{2N}(0)\rangle$, which displays a power-law decay with a *universal* exponent: $\mathcal{R}_N(x) \sim x^{-2N(N-1)/(N+2)}$. For larger N, the phase transition is nonuniversal. For $N \geq 4$, a strongly relevant perturbation is indeed generated, which takes the form of the second thermal operator ϵ_2 of the $\mathbb{Z}_N$ CFT with scaling dimension $12/(N+2)$ [9]. The resulting model is integrable and the transition is either of first-order or in the U(1) universality class depending on N and the sign of the coupling constant of ϵ_2 [15]. In the $N = 4$ case, i.e., a special case of the Ashkin-Teller model, the $\mathbb{Z}_4$ criticality can emerge with the introduction of the interaction $\lambda \sum_i (P^\dagger_{00,i}P_{00,i})^2$ which can eliminate the operator ϵ_2 by a fine-tuning of λ.

Mott phases.—At the commensurate $1/2N$ filling, i.e., one atom per site, an umklapp term $\sim \cos(\sqrt{8\pi N}\Phi)$ is generated in the density sector and becomes relevant when $K_d < 1/N$ leading to the opening of a density gap. We further distinguish between three different Mott phases [16]. The first one lies in the SDW region I and is qualitatively similar to the one encountered in the pure SU($2N$) Hubbard chain [13]. The two others have a spin gap and can be distinguished with respect to the confinement properties of atomic currents. In the region II we find for large enough V a $2k_F$-ordered ADW and spin-Peierls ordering with a $2N$ ground-state degeneracy. In the region III the crossover line between BCS and MDW phases identifies to the Mott transition line. At filling $1/2N$, while the BCS phase remains, the MDW regime locks into a $2Nk_F$ MDW and displays a dimerized bond ordering for all N. This Mott phase is unusual since there is no one-particle density long-range fluctuation due to the confinement of atomic currents. Remarkably enough, we find that this MDW Mott phase is the only gapped phase directly connected to the BCS superfluid phase.

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