Anti–de Sitter-Space/Conformal-Field-Theory Casimir Energy for Rotating Black Holes

G. W. Gibbons,[1] M. J. Perry,[1] and C. N. Pope[2]

[1]DAMTP, Centre for Mathematical Sciences, Cambridge University, Wilberforce Road, Cambridge CB3 OWA, United Kingdom

[2]George P. & Cynthia W. Mitchell Institute for Fundamental Physics, Texas A&M University, College Station, Texas 77843-4242, USA

(Received 9 August 2005; published 28 November 2005)

We show that, if one chooses the Einstein static universe as the metric on the conformal boundary of Kerr–anti-de Sitter spacetime, then the Casimir energy of the boundary conformal field theory can easily be determined. The result is independent of the rotation parameters, and the total boundary energy then straightforwardly obeys the first law of thermodynamics. Other choices for the metric on the conformal boundary will give different, more complicated, results. As an application, we calculate the Casimir energy for free self-dual tensor multiplets in six dimensions and compare it with that of the seven-dimensional supergravity dual. They differ by a factor of 5/4.

DOI: 10.1103/PhysRevLett.95.231601 PACS numbers: 11.25.Tq, 04.70.Dy, 98.80.Cq

An interesting application of the anti-de Sitter/conformal field theory (AdS/CFT) correspondence [1–3] is to consider for the bulk solution a rotating Kerr–anti-de Sitter spacetime. As was discussed in Ref. [4], the boundary theory in this case should describe a conformal field theory in a rotating Einstein universe, allowing one, in principle, to study the effects of rapid rotation on the thermodynamics of the system. Of particular interest is the behavior of the CFT on the boundary as the rotation parameters achieve their maximal values consistent with the nonexistence of closed timelike curves. The chronology protection conjecture of Hawking [5] suggests that going beyond this limit should be physically impossible. This may be associated with a divergence of the free energy of the CFT as one approaches the limit and with a possible nonunitarity of the CFT if one passes beyond it.

The study of the thermodynamics of such rotating systems is quite involved and subtle, both in the bulk and in the passage to the boundary theory. In the bulk, there are subtleties concerning the definition of the energy, or mass, of the rotating AdS black hole. As we showed in Ref. [6], it is important that one evaluate the energy and the angular velocities with respect to a frame that is asymptotically nonrotating at infinity, in order to obtain quantities that satisfy the first law of thermodynamics,

$$dE = TdS + \Omega^i dJ_i. \tag{1}$$

In particular, a commonly considered frame that rotates relative to the asymptotically static frame, and which arose when the Kerr–AdS metrics were first constructed in Boyer-Lindquist type coordinates, is particularly inappropriate for defining the energy and angular velocities, since its asymptotic rotation rate is dependent on the angular-momentum parameters of the metric [7].

In Ref. [7], we addressed the question of how one should map between the bulk thermodynamic variables and the corresponding variables on the boundary. We showed that, using the standard UV/IR connection, the bulk variables (E, T, S, Ω^i, J_i) that satisfy the first law as in (1) map straightforwardly into boundary variables (E, T, S, ω^i, j_i) that also satisfy the first law, now with the addition of a pressure term,

$$de = tds + \omega^i dj_i - pdv. \tag{2}$$

This mapping is implemented, for an n-dimensional bulk spacetime, by imposing the relations

$$\begin{aligned} &e = \frac{l}{y}E, \qquad \omega^i = \frac{l}{y}\Omega^i, \qquad t = \frac{l}{y}T, \qquad s = S, \\ &j_i = J_i, \qquad v = \mathcal{A}_{n-2}y^{n-2}, \qquad p = \frac{e}{(n-2)v}. \end{aligned} \tag{3}$$

Here l is the radius of the asymptotically AdS spacetime, in the sense that $R_{\mu\nu} = -(n-1)l^2 g_{\mu\nu}$, $\mathcal{A}_{n-2}$ is the volume of the unit $(n-2)$ sphere, and y is the radius of a large sphere, with round spherical metric, near the boundary in AdS. As we emphasized in Ref. [7], the natural choice of boundary metric is that of the Einstein static universe, ESU_{n-1}, i.e., the standard product metric on $\mathbb{R} \times S^{n-2}$. In other words, one introduces a set of coordinates in which the Kerr–AdS metric approaches the canonical AdS_n metric at infinity, in the form

$$d\bar{s}_n^2 = -(1 + y^2 l^{-2})dt^2 + \frac{dy^2}{1 + y^2 l^{-2}} + y^2 d\Omega_{n-2}^2, \tag{4}$$

where $d\Omega_{n-2}^2$ is the metric on the unit $(n-2)$ sphere. The conformal boundary is then located at $y \to \infty$, and the induced metric has the standard ESU_{n-1} form

$$d\bar{s}_{n-1}^2 = -dt^2 + l^2 d\Omega_{n-2}^2. \tag{5}$$

More precisely, in an AdS-type conformal compactification, the physical metric is given by $g = \Omega^{-2}\tilde{g}$, where $\Omega = 0$ and $d\Omega \neq 0$ on the timelike conformal boundary $\mathcal{I}$, with $\tilde{g}$ being nonsingular in a neighborhood of $\mathcal{I}$. The metric induced on the boundary $\mathcal{I}$ is $h = \tilde{g}|_{\mathcal{I}}$ and depends on the choice of the conformal factor Ω. If $\Omega \to f\Omega$, for some function f which is nonvanishing in a neighborhood of $\mathcal{I}$, then the boundary metric undergoes a conformal rescaling $h \to f^2 h$. In order to obtain the standard metric on ESU_{n-1}, one simply chooses $\Omega = \frac{l}{y}$.

Our result in Ref. [7] refuted a recent surprising claim in Ref. [8], where it was asserted that, in order to get thermodynamic quantities that satisfied the first law on the boundary, it was necessary to start from thermodynamic quantities in the bulk that did not satisfy the first law (in fact, the quantities defined relative to the rotating frame we mentioned earlier). As we showed, the puzzling results in Ref. [8] were associated with the use of a somewhat unnatural conformal factor Ω and, hence, a complicated boundary metric h whose spatial sections were not round spheres and which was not ultrastatic; i.e., g_{tt} depended on spatial position.

In Ref. [7], our principal interest was in the properties and thermodynamics of the bulk theories, and so it was not relevant to consider the contribution of Casimir energies on the boundary. Such terms do play an important rôle in the boundary CFT, and much work has been done on calculating them. For the case of Schwarzschild-AdS spacetime, the boundary Casimir contribution was evaluated in Ref. [9]. Casimir calculations have also been performed for the case of rotating Kerr–AdS black holes, in Ref. [10] and, more recently, in Ref. [11]. The result obtained in these papers for the four-dimensional boundary theory is

$$E_{\mathrm{Casimir}} = \frac{3\pi^2 l^2}{4\kappa^2}\left(1 + \frac{(\Xi_a - \Xi_b)^2}{9\Xi_a \Xi_b}\right), \tag{6}$$

where $\kappa^2/(8\pi)$ is Newton's constant, $\Xi_a = 1 - a^2 l^{-2}$, $\Xi_b = 1 - b^2 l^{-2}$, and a and b are the rotation parameters of the five-dimensional Kerr–AdS black hole given in Ref. [4].

The expression (6) for the Casimir energy of the boundary is a somewhat surprising result, since it depends on the rotation parameters a and b of the Kerr–AdS black hole. As we have argued above, the most natural conformal frame in which to formulate the boundary CFT is one in which the metric approaches the form (4), and the boundary metric is that of ESU_{n-1}, and, since this metric is manifestly independent of a and b, the Casimir energy will necessarily also be independent of a and b. Evidently, therefore, the conformal boundary metric chosen in Refs. [10,11] is not the one we are advocating. In the remainder of this Letter, we shall endeavour to convince the reader that our proposed choice of conformal boundary metric for the CFT dual to the Kerr–AdS metric is by far the most natural one and that it has the very satisfactory feature that it leads to a genuinely constant Casimir energy in the boundary theory.

In fact, in order to demonstrate our point, we need only collect a few results from previous papers. We shall discuss mostly the general n-dimensional case, since it is just as easy to discuss it generally as in any specific dimension. The general Kerr–AdS metrics were obtained in Refs. [12,13]. The metrics have $N \equiv [(n-1)/2]$ independent rotation parameters a_i in N orthogonal 2-planes. We have $n = 2N + 1$, when n is odd, and $n = 2N + 2$, when n is even. The metrics can be described by introducing N azimuthal angles ϕ_i and $(N + \epsilon)$ "direction cosines" μ_i obeying the constraint

$$\sum_{i=1}^{N+\epsilon} \mu_i^2 = 1, \tag{7}$$

where $\epsilon = (n-1) \bmod 2$. In Boyer-Lindquist type coordinates that are asymptotically nonrotating, the metrics are given by [12,13]

$$\begin{aligned} ds_n^2 = {}& -W(1 + r^2 l^{-2})dt^2 + \sum_{i=1}^{N} \frac{r^2 + a_i^2}{\Xi_i}\mu_i^2 d\varphi_i^2 \\ & + \sum_{i=1}^{N+\epsilon} \frac{r^2 + a_i^2}{\Xi_i} d\mu_i^2 + \frac{U dr^2}{V - 2m} \\ & + \frac{2m}{U}\left(W dt - \sum_{i=1}^{N} \frac{a_i \mu_i^2 d\varphi_i}{\Xi_i}\right)^2 \\ & - \frac{l^{-2}}{W(1 + r^2 l^{-2})}\left(\sum_{i=1}^{N+\epsilon} \frac{r^2 + a_i^2}{\Xi_i}\mu_i d\mu_i\right)^2, \end{aligned} \tag{8}$$

where

$$\begin{aligned} W &\equiv \sum_{i=1}^{N+\epsilon} \frac{\mu_i^2}{\Xi_i}, \qquad \Xi_i \equiv 1 - a_i^2 l^{-2}, \\ U &\equiv r^{\epsilon} \sum_{i=1}^{N+\epsilon} \frac{\mu_i^2}{r^2 + a_i^2} \prod_{j=1}^{N} (r^2 + a_j^2), \end{aligned} \tag{9}$$

$$V \equiv r^{\epsilon - 2}(1 + r^2 l^{-2}) \prod_{i=1}^{N} (r^2 + a_i^2), \tag{10}$$

and it is understood, in the even-dimensional case, that $a_{N+1} = 0$. The metrics satisfy $R_{\mu\nu} = -(n-1)l^{-2} g_{\mu\nu}$. The constant-$r$ spatial surfaces at large distance are inhomogeneously distorted $(n-2)$ spheres. Making the coordinate transformations

$$\Xi_i y^2 \hat{\mu}_i^2 = (r^2 + a_i^2)\mu_i^2, \tag{11}$$

where $\sum_i \hat{\mu}_i^2 = 1$, the metrics at large y approach the standard AdS form given in (4), where

$$d\Omega_{n-2}^2 = \sum_{k=1}^{N+\epsilon} d\hat{\mu}_k^2 + \sum_{k=1}^{N} \hat{\mu}_k^2 d\varphi_k^2, \tag{12}$$

with round $(n-2)$ spheres of volume $\mathcal{A}_{n-2} y^{n-2}$ at radius y, where $\mathcal{A}_{n-2}$ is the volume of the unit $(n-2)$ sphere. Note, in particular, that the boundary metric does not depend on any of the black-hole parameters.

The boundary CFT will be defined on the surface $y =$ constant at very large y. The Casimir energy in the boundary theory is clearly independent of the mass m of the Kerr–AdS black hole, since the boundary metric does not depend upon m. Therefore, for convenience, we can evaluate it by first setting $m = 0$ in (8), which implies that the metric becomes purely AdS itself, in an unusual coordinate system. As was shown in Refs. [12,13], if one now performs the coordinate transformation (11), the AdS metric becomes *exactly* the canonical metric given in (4). The calculation of the Casimir energy for the boundary of Kerr–AdS is, therefore, reduced to the standard calculation for the Einstein static universe, ESU_{n-1}. The answer is obviously independent of the rotation parameters, since they do not appear in the boundary metric.

One may calculate the Casimir energy in a number of ways. In Ref. [9], it was shown that the use of the holographic stress tensor for the conformal anomaly for four-dimensional $\mathcal{N} = 4$ super-Yang-Mills gives

$$E_{\mathrm{Casimir}} = \frac{3\pi^2 l^2}{4\kappa^2}. \tag{13}$$

This value agrees with a direct calculation of the Casimir energy using zeta-function regularization of the sums over energy eigenvalues for $6N^2$ scalar fields, $4N^2$ Weyl spinor fields, and N^2 vector fields on S^3. For our choice of conformal boundary metric, (13) is therefore the Casimir energy in the four-dimensional boundary theory dual to the five-dimensional Kerr–AdS metric. Combined with our calculation of the bulk energy term obtained in Ref. [6], the complete CFT energy is given by

$$E_{\mathrm{tot}} = \frac{2\pi^2 m(2\Xi_a + 2\Xi_b - \Xi_a\Xi_b)}{\kappa^2 \Xi_a^2 \Xi_b^2} + \frac{3\pi^2 l^2}{4\kappa^2}. \tag{14}$$

The calculations in any other dimension proceed similarly, again yielding Casimir energies that are necessarily independent of the black-hole rotation parameters. For example, in the case of a seven-dimensional Kerr–AdS bulk spacetime, we obtain a total boundary energy given by

$$E_{\mathrm{tot}} = \frac{2m\pi^3}{\kappa^2(\prod_i \Xi_i)}\left(\frac{1}{\Xi_1} + \frac{1}{\Xi_2} + \frac{1}{\Xi_3} - \frac{1}{2}\right) - \frac{5\pi^3 l^4}{16\kappa^2}, \tag{15}$$

where the first term is the bulk energy that we calculated in Ref. [6], and we have read off the Casimir term by setting $a = 0$ in Eq. (51) of Ref. [10]. This value, which came from the use of the holographic stress tensor, should be compared with the value obtained directly by zeta-function regularization of the sums over energy eigenvalues of $20N^3$ scalars, $8N^3$ Weyl fermions, and $4N^3$ self-dual 3-forms, making up $4N^3$ copies [14] of the $(2, 0)$ tensor multiplet. Using the energies and degeneracies tabulated in Ref. [15], we find that the Casimir energy for N_0 scalars, $N_{1/2}$ Weyl fermions, and N_T self-dual 3-forms is

$$E_{\mathrm{Casimir}} = -\frac{(124N_0 + 1835N_{1/2} + 11\,460N_T)}{241\,920 l}. \tag{16}$$

For a $4N^3$ multiplet, with the seven-dimensional AdS/CFT relation $N^3 = 3\pi^3 l^5/(2\kappa^2)$, this gives

$$E_{\mathrm{Casimir}} = -\frac{25\pi^3 l^4}{64\kappa^2}. \tag{17}$$

Clearly, this does not agree with the Casimir term in (15); the ratio is in fact $5/4$, which is not the same as the $4/7$ ratio conjectured in Ref. [10]. Presumably, these differences reflect the absence of the nonrenormalization theorems that appear to account for the agreement of the various methods in the four-dimensional case.

We have demonstrated that, by choosing the conformal boundary metric defined by taking $y =$ constant, we obtain a very simple framework for describing the thermodynamics of the boundary field theory, with a very simple expression for the Casimir contribution to the energy, which is independent of the parameters in the Kerr–AdS metric. In particular, this means that the first law of thermodynamics continues to hold in a straightforward manner, when the Casimir energy is included. This contrasts with the more complicated situation in the case of the conformal boundary metric chosen in Refs. [10,11], where, as was shown in Ref. [11], an additional diffeomorphism term must be included in the first law in order to compensate for the dependence of the boundary metric on the rotation parameters of the black hole.

The Boyer-Lindquist form of the boundary metric (in nonrotating coordinates) is obtained by taking the limit $r \to \infty$ of $(l^2/r^2)ds^2$, where ds^2 is given by (8). This yields

$$ds_{n-1}^2 = -Wdt^2 + l^2 \sum_{i=1}^{N+\epsilon} \frac{1}{\Xi_i} d\mu_i^2 + l^2 \sum_{i=1}^{N} \frac{1}{\Xi_i} \mu_i^2 d\varphi_i^2 - \frac{l^2}{W}\left(\sum_{i=1}^{N+\epsilon} \frac{\mu_i d\mu_i}{\Xi_i}\right)^2. \tag{18}$$

If we introduce new coordinates $\hat{\mu}_i$, satisfying $\sum_i \hat{\mu}_i^2 = 1$, by the transformations

$$\mu_i = \sqrt{W\Xi_i}\hat{\mu}_i, \tag{19}$$

then the metric (18) becomes

$$ds_{n-1}^2 = Wd\bar{s}_{n-1}^2, \tag{20}$$

where $d\bar{s}_{n-1}^2$ is the standard metric on ESU_{n-1}, given by (5) and (12), with W expressed in terms of the $\hat{\mu}_i$ as $W = \sum_i \Xi_i \hat{\mu}_i^2$. This is the generalization to arbitrary dimension of the five-dimensional demonstration in Ref. [6] that the $r =$ constant "Boyer-Lindquist" boundary metric (18) and the $y =$ constant Einstein static universe boundary metric (5) are related by a Weyl rescaling, and, thus, they lie in the same conformal class. When considering the first law, one varies the rotation parameters a_i. If one uses the Boyer-Lindquist boundary metric (18), this variation induces an infinitesimal change of the conformal factor and, hence, a change of the Casimir contribution to the energy, as described in detail in five dimensions in Ref. [11] [see their Eqs. (6.51)–(6.54)].

We are not, of course, saying that the more complicated results for the Casimir energies obtained in Refs. [10,11] are wrong but, rather, that they are the consequence of having made a less felicitous choice for the conformal boundary metric or, equivalently, the conformal factor. As one can see from the calculations presented in Ref. [11], although a coordinate transformation was performed in order to simplify the conformal boundary metric, it did not lead to as great a simplification as can be achieved by using the transformation (11).

We shall conclude with a remark on the action of the asymptotic symmetry group. Since we are working on the universal covering space $\widetilde{\mathrm{AdS}}_n$ of $\mathrm{AdS}_n \equiv SO(n-1,2)/SO(n-2,1)$, this is an infinite covering $\widetilde{SO}(n-1,2)$ of $SO(n-1,2)$. The bulk energy E and the angular momenta J_i transform properly under the action of the asymptotic symmetry group as the components of a 5×5 antisymmetric tensor J_{AB}, which may be thought of as an element of the Lie algebra $\widetilde{\mathfrak{so}}(n-1,2)$. The transformation rule is just the adjoint action.

The transformation properties of the Casimir energy are more subtle. In general, the asymptotic group acts on the boundary by consometries, i.e., preserving the boundary metric h only up to a conformal factor. Depending upon one's choice of the boundary metric, i.e., of the representative in its conformal equivalence class, there may be a subgroup which acts by isometries. If, as we have done, we choose the Einstein static universe ESU_{n-1} as boundary metric, then this subgroup is $\mathbb{R} \times SO(n-1)$ and is maximal. Indeed, it is an infinite covering of the maximal compact subgroup of $SO(n-1,2)$. The Casimir contribution to the energy is clearly invariant under the subgroup of isometries of the boundary metric, but it transforms in a well-defined but more complicated and nontrivial fashion under those elements of the asymptotic symmetry group which do not induce isometries. Choosing the Einstein static universe ESU_{n-1} as boundary metric minimizes these complications.

In summary, we have seen from previous work that the description of the bulk thermodynamics of rotating black holes in an AdS background is greatly simplified if one refers the energy and the angular velocities of the black hole to a coordinate frame that is nonrotating at infinity. Especially, one should not choose a rotating coordinate frame whose asymptotic angular velocity depends upon the parameters of the black hole. In this Letter, we have furthermore shown that the description of the boundary CFT is greatly simplified if one likewise defines a conformal boundary metric that does not depend upon the parameters of the black hole. This is straightforwardly achieved by applying the coordinate transformation (11) to the asymptotically static form of the Kerr–AdS metrics given in (8) and defining the boundary metric as the section $y =$ constant as y tends to infinity. By this means, all the complications associated with the unnecessary introduction of black-hole parameter dependence of the thermodynamical quantities in the boundary theory are avoided.

We thank Adel Awad, David Berman, Clifford Johnson, and Ioannis Papadimitriou for discussions. C. N. P. thanks the Relativity and Cosmology group, Cambridge, for hospitality during the course of this work. Research was supported in part by DOE Grant No. DE-FG03-95ER40917 and NSF Grant No. INTO3-24081.

[1] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998).
[2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998).
[3] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).
[4] S. W. Hawking, C. J. Hunter, and M. M. Taylor-Robinson, Phys. Rev. D **59**, 064005 (1999).
[5] S. W. Hawking, Phys. Rev. D **46**, 603 (1992).
[6] G. W. Gibbons, M. J. Perry, and C. N. Pope, Classical Quantum Gravity **22**, 1503 (2005).
[7] G. W. Gibbons, M. J. Perry, and C. N. Pope, Phys. Rev. D **72**, 084028 (2005).
[8] R. G. Cai, L. M. Cao, and D. W. Pang, Phys. Rev. D **72**, 044009 (2005).
[9] V. Balasubramanian and P. Kraus, Commun. Math. Phys. **208**, 413 (1999).
[10] A. M. Awad and C. V. Johnson, Phys. Rev. D **63**, 124023 (2001).
[11] I. Papadimitriou and K. Skenderis, J. High Energy Phys. 08 (2004) 004.
[12] G. W. Gibbons, H. Lü, D. N. Page, and C. N. Pope, J. Geom. Phys. **53**, 49 (2005).
[13] G. W. Gibbons, H. Lü, D. N. Page, and C. N. Pope, Phys. Rev. Lett. **93**, 171102 (2004).
[14] F. Bastianelli, S. Frolov, and A. A. Tseytlin, J. High Energy Phys. 02 (2000) 013.
[15] D. Kutasov and F. Larsen, J. High Energy Phys. 01 (2001) 001.