Measurement-Based Teleportation along Quantum Spin Chains

J. P. Barjaktarevic,[1,*] R. H. McKenzie,[1] J. Links,[2] and G. J. Milburn[3]
[1]*Department of Physics, University of Queensland, Queensland QLD 4072, Australia*
[2]*Department of Mathematics, University of Queensland, Queensland QLD 4072, Australia*
[3]*Center for Quantum Computer Technology, School of Physical Sciences, The University of Queensland, QLD 4072 Australia*
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We examine the teleportation of an unknown spin-1/2 quantum state along a quantum spin chain with an even number of sites. Our protocol, using a sequence of Bell measurements, may be viewed as an iterated version of the 2-qubit protocol of C. H. Bennett *et al.* [Phys. Rev. Lett. **70**, 1895 (1993)]. A decomposition of the Hilbert space of the spin chain into 4 vector spaces, called Bell subspaces, is given. It is established that any state from a Bell subspace may be used as a channel to perform unit fidelity teleportation. The space of all spin-0 many-body states, which includes the ground states of many known antiferromagnetic systems, belongs to a common Bell subspace. A channel-dependent teleportation parameter $\mathcal{O}$ is introduced, and a bound on the teleportation fidelity is given in terms of $\mathcal{O}$.

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Quantum many-body spin Hamiltonians often have entangled ground states, and an understanding of entanglement may in turn lead to a greater understanding of the physics of many-body quantum systems with strong correlations [1]. Although there exist several proposed measures for multipartite entanglement [2–4], there is no known definitive measure which significantly aids this understanding. In this Letter, we instead characterize the entanglement of multipartite states by one of the manifest properties of entanglement, the ability to perform teleportation. This is investigated using a protocol involving successive Bell measurements. Specifically, we construct a teleportation parameter, which is a function of the channel state, that bounds the teleportation fidelity. We then examine the ground states of several known quantum spin chain Hamiltonians and establish that they provide channels capable of performing perfect fidelity teleportation, for an arbitrary length of the chain.

Teleportation over a Bell state by Bell-basis measurement is well understood[5]. We denote our two-dimensional spin space as V, with basis $|\uparrow\rangle, |\downarrow\rangle$. Let v^i, $i = 1, 2, 3, 4$ represent the standard Bell-basis states, where v^0 is the singlet $v^0 = (|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)/\sqrt{2}$. The other Bell states, v^i, may be written as $v^i = (I \otimes X^i)v^0$, where $X^0 = \sigma_I$, $X^1 = \sigma_x$, $X^2 = -\sigma_z$, and $X^3 = i\sigma_y$, and the σ matrices are the usual Pauli matrices for spin-1/2 particles. These states are simultaneous eigenstates of $\sigma^x \otimes \sigma^x$ and $\sigma^z \otimes \sigma^z$. Let us introduce the notation $|kl\}$ for Bell states such that $\sigma^x \otimes \sigma^x|kl\} = k|kl\}$, and $\sigma^z \otimes \sigma^z|kl\} = l|kl\}$ [6], where $k, l = \pm 1$.

The standard scheme for teleportation proceeds as follows [5]. Let $P^{|jk\}}$ denote the projections onto the state $|jk\}$. Given a target state which we wish to teleport, $|v\rangle$, and a two-particle Bell state $|jk\}$ over which we wish to teleport $|v\rangle$, we simply perform a Bell-basis measurement across the target state, and half of the two-particle state, as in Fig. 1(a). We can understand this result by making the following decomposition:

$$|v\rangle \otimes |jk\} = \frac{1}{2}\sum_{p,q} |pq\} \otimes X^{jk}_{pq}|v\rangle, \tag{1}$$

where X^{jk}_{pq} is one of the matrices X^i, as in Table I. We can identify X^{jk}_{pq} with a unitary correction, which we must invert to yield the target state $|v\rangle$.

Multipartite channel teleportation.—We now wish to find multipartite channels which are amenable to a teleportation protocol. We propose a scheme in which a series of Bell measurements are made along a chain, followed by some unitary operation on the last site, which will be conditioned on the results of the Bell measurements. We will consider channel states of an even number $L = 2\mathcal{L}$ of spin-1/2 particles. We define the L-particle state $|\vec{j}\vec{k}\} = |j_1k_1\} \otimes \ldots \otimes |j_\mathcal{L}k_\mathcal{L}\}$, where $\vec{j} = (j_1, j_2, \ldots, j_\mathcal{L})$, $\vec{k} = (k_1, k_2, \ldots, k_\mathcal{L})$ are $\mathcal{L}$-dimensional vectors. A funda-

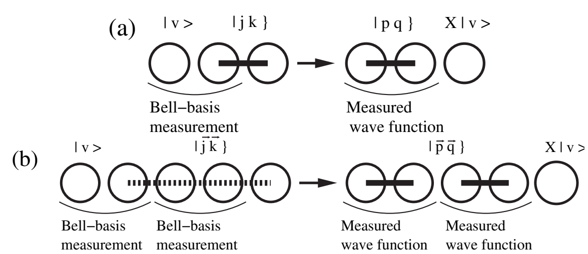

FIG. 1. A schematic representation of the proposed teleportation protocol, showing input state $|v\rangle$ and final state $X|v\rangle$. (a) A Bell-basis measurement over a single pair of maximally entangled spins, as in Eq. (1), represented by a solid line between two sites. (b) A schematic representation of our protocol for a general spin quantum state, as in Eq. (2), with nontrivial entanglement, represented by the dashed line. Here we assume that the state $|\vec{j}\vec{k}\}$ lies within a Bell subspace, yielding a pure output state, $X|v\rangle$. X is known from the measurement outcome, as per Table I.

TABLE I. Unitary correction, X^{jk}_{pq}, to be inverted on the final qubit, where we start with a quantum state $|v\rangle \otimes |jk\}$ and project onto $|pq\}$ with $P^{|pq\}} = |pq\}\{pq| \otimes I$. These correspond to either no correction (X^0), a bit flip (X^1), a phase flip (X^2), or both a bit and a phase flip (X^3).

$P^{\|pq\}}$ \ $\|jk\}$	$\|--\}$	$\|-+\}$	$\|+-\}$	$\|++\}$
$P^{\|--\}}$	$-X^0$	$-X^1$	$-X^2$	$-X^3$
$P^{\|-+\}}$	X^1	X^0	$-X^3$	$-X^2$
$P^{\|+-\}}$	X^2	X^3	X^0	X^1
$P^{\|++\}}$	$-X^3$	$-X^2$	X^1	X^0

mental result is that we can make the following decomposition [7]

$$|v\rangle \otimes |\vec{j}\,\vec{k}\} = \frac{1}{2^{\mathcal{L}}} \sum_{\vec{p},\vec{q}} |\vec{p}\,\vec{q}\} \otimes X^{j_{\mathcal{L}}k_L}_{p_{\mathcal{L}}q_{\mathcal{L}}} \ldots X^{j_1k_1}_{p_1q_1}|v\rangle. \quad (2)$$

Since the corrections are closed under composition, we may write $X^{j_{\mathcal{L}}k_{\mathcal{L}}}_{p_{\mathcal{L}}q_{\mathcal{L}}} \ldots X^{j_1k_1}_{p_1q_1} = X^{\vec{j}\,\vec{k}}_{\vec{p}\,\vec{q}} = \pm X^i$ for some i.

Physically, each component $|pq\}$ of the decomposition corresponds to a history of Bell measurements, which gives some unitary correction to the target state in the final site. Thus we can understand our protocol as an iterated version of the Bennett *et al.* scheme. However, the current work represents a fundamental extension to general multipartite channels.

It can be shown [7] that states $|\vec{j}\,\vec{k}\}$ are simultaneous eigenstates of the following operators

$$\mathcal{O}_\alpha = \otimes^L_{i=1}\sigma^\alpha_i,\ \alpha = x, y, z \quad (3)$$

with eigenvalues $\prod_i j_i$, $(-1)^{\mathcal{L}}\prod_i j_i k_i$, and $\prod_i k_i$, respectively. We can decompose the total L particle Hilbert space, $V^{\otimes L}$, into four subspaces as

$$V^{\otimes L} = V^L_{[--]} \oplus V^L_{[-+]} \oplus V^L_{[+-]} \oplus V^L_{[++]}, \quad (4)$$

where $V^L_{[jk]}$ is the subspace of eigenstates of $\mathcal{O}_x$ and $\mathcal{O}_z$ with eigenvalues j and k, respectively.

We refer to the 4 subspaces as Bell subspaces since they reduce to the Bell basis in the two-qubit case. For any state within a given subspace $V^L_{[kl]}$, due to the linearity of Eq. (2), we find that we are able to perform unit fidelity teleportation with corrections which are in 1-1 correspondence with the assignments given in Table I. It is vital to note that because these subspaces exist, for $\mathcal{L} > 1$ there are infinitely many channel states which we may use to implement this scheme. This is in stark contrast to the 2-qubit channel case where there are only four perfect channels. Further details can be found in a longer companion paper [7].

Fidelity.—Given a channel quantum state, $|\psi\rangle$, which we believe to belong to the subspace $V^L_{[pq]}$, we perform Bell-basis measurements along the chain, and then apply the appropriate correction to the last site. Let us decompose our channel state as

$$|\psi\rangle = \sum_{i,j=\pm} c_{i,j}|\phi_{i,j}\rangle, \quad (5)$$

where $|\phi_{i,j}\rangle$ are orthonormal vectors, each lying in subspaces $V^L_{[ij]}$. The condition that $|\psi\rangle$ lies within the subspace $V^L_{[pq]}$ implies $c_{i,j} = \delta_{i,p}\delta_{j,q}$, which might only be approximately satisfied for real quantum states. Projecting onto the Bell basis yields the measurement result $|\vec{a}\,\vec{b}\}$ with probability, $p_{|\vec{a}\,\vec{b}\}}$. If the target quantum state is $|v\rangle$, then the resulting quantum state will be $|\vec{a}\,\vec{b}\} \otimes |v'\rangle$ where $|v'_{\vec{a}\,\vec{b}}\rangle \propto \sum_{i,j=\pm} c_{i,j}X^{ij}_{\vec{a}\,\vec{b}}|v\rangle$ is a superposition of different corrections onto $|v\rangle$. The fidelity for a particular state $|v\rangle$ is given by $\mathcal{F} = |\langle v'_{\vec{a}\,\vec{b}}|X^{pq}_{ab}|v\rangle|^2$.

We now define a quantity which we will use to characterize the fidelity of such a teleportation scheme. We may use such a property to assess the feasibility of a teleportation scheme. For any quantum spin state $|\psi\rangle$, we define the *teleportation parameter* as

$$\mathcal{O} = \sum_{\alpha=x,y,z} |\langle\psi|\mathcal{O}_\alpha|\psi\rangle|, \quad (6)$$

where $\mathcal{O}_\alpha$ is given by Eq. (3). We note that $\mathcal{O}$ is related to nonlocal correlations in $|\psi\rangle$, bearing some similarity to a string order parameter [8]. The minimum possible fidelity given perfect Bell-basis measurements is a monotonically increasing function of $\mathcal{O}$ [7], as seen in Fig. 2. Note that the fidelity $\mathcal{F}$ is not uniquely defined by $\mathcal{O}$, but is also dependent on $|v\rangle$ and the measurement history.

It can be shown [7] that $\mathcal{O} = 4|c_{p,q}|^2 - 1$, such that when $\mathcal{O}$ is close to 3, we satisfy the condition $c_{i,j} \simeq \delta_{i,p}\delta_{j,q}$, with equality when $\mathcal{O} = 3$. One can prove that $\mathcal{O}$ can be used to give a lower bound on the fidelity of

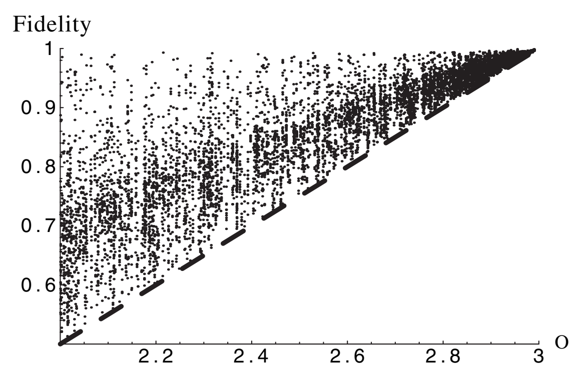

FIG. 2. Data points correspond to the fidelity of teleportation for a range of target states and channels. The fidelity is shown against the quantity $\mathcal{O}$ as defined in Eq. (6) for 2000 4-qubit channels. The dashed line is a lower bound for these points and is given by the inequality in Eq. (7). This clearly shows the tendency $\mathcal{F} \to 1$ as $\mathcal{O} \to 3$.

teleportation through $|\psi\rangle$, by the following inequality [7]:

$$\mathcal{F} \geq \frac{\mathcal{O} - 1}{2}. \tag{7}$$

A crucial point is that the ability to perform long distance, high fidelity teleportation is encompassed in the inequality Eq. (7). In order to assess the feasibility of a specific implementation of this teleportation protocol, one simply needs to ensure that $\mathcal{O}$ is appropriately close to 3.

Cluster state QC.—The protocol we have presented is based purely upon measurement, classical communication, and local unitary operations. These operations are sufficient for universal quantum computation (QC) [9–11], particularly as manifest in cluster state QC. Hence, characterizing the ability of a system to perform high fidelity teleportation is useful for the purposes of QC.

By inspection, the regular 2-site cluster state [10], $|\downarrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle - |\uparrow\uparrow\rangle$ does not lie in any of the Bell subspaces, since it is a linear superposition of two Bell states. However, we may rewrite this state as $|\downarrow\rightarrow\rangle + |\uparrow\leftarrow\rangle$, where $|\rightarrow\rangle = 1/\sqrt{2}(|\downarrow\rangle + |\uparrow\rangle)$ and $|\leftarrow\rangle = 1/\sqrt{2}(|\downarrow\rangle - |\uparrow\rangle)$ are the usual eigenvectors of σ_x. This state now resembles a Bell state in the z, x basis rather than the z, z basis. More generally, it can be shown that the one-dimensional L-qubit cluster states of Ref. [10] can be mapped to the subspace $V^L_{[++]}$ through a sequence of local unitary transformations [7]. This may be expected, as our scheme resembles cluster state methods in the use of local measurement and feed forward. The main difference is in the preparation. Cluster states require an initialization, in which entanglement is provided by a series of controlled-phase entangling gates, while in our scheme, we require a state to lie in a Bell subspace.

Spin-1/2 models.—The decomposition of V^L into 4 subspaces with different corrections turns out to be extremely useful in understanding systems which have ground states lying entirely in one of these subspaces. We present several model Hamiltonians for which this is true. It is interesting to note that all of these ground states are similar to spin liquid states[12], since every member of $V^L_{[kl]}$ has every 1-qubit reduced density matrix equal to $I/2$ [7]. Hence, each site is also maximally entangled with the rest of the chain. Any state in one of these subspaces has maximum localizable entanglement[13] with respect to Bell measurements.

A family of nearest- and next-nearest neighbor antiferromagnetic spin exchange Hamiltonians is parametrized with $\beta > 0$ by

$$H = \sum_{i=1}^{N} \hat{S}_i.\hat{S}_{i+1} + \beta \hat{S}_i.\hat{S}_{i+2}, \tag{8}$$

for which $\beta = 1/2$ yields the Majumdar-Ghosh Hamiltonian [14]. The ground state of this Hamiltonian is simply comprised of a tensor product of singlets, $\otimes_{k=1}^{N/2}|v^0\rangle$, and one may use this to perform unit fidelity teleportation with repeated Bell-basis measurements along the chain. However, it is possible to show that the ground state of Eq. (8) for any value of $\beta > 0$ lies within a Bell subspace, including the specific case of $\beta = 0$, which corresponds to the Heisenberg Hamiltonian.

More generally, it can be shown rigorously that any quantum state with a total spin 0 lies within one of the Bell subspaces, $V^L_{[kl]}$, which we outline as follows. Given $\Psi^0 = v^0 \otimes \ldots \otimes v^0$, it is clear that Ψ^0 must have total spin 0. Further, any permutation of sites in Ψ^0 must also have spin 0, and belong to the same subspace [15]. In fact, it is possible to decompose every spin-0 state as a sum over permutations of the sites of Ψ^0 [16]. This is known in the chemistry literature as resonant valence bond theory [17].

It is interesting to note that for the antiferromagnetic Ising Hamiltonian [18], there are two degenerate product form ground states, $|\Phi^-\rangle = |\uparrow\downarrow \ldots \downarrow\rangle$ and $|\Phi^+\rangle = |\downarrow\uparrow \ldots \uparrow\rangle$. The decomposition of these will have equal support in more than one Bell subspace:

$$|\Phi^\pm\rangle = (|+-\} \pm |--\}) \otimes \ldots \otimes (|+-\} \pm |--\}).$$

However, the superposition, $|\Phi^0\rangle = (|\Phi^+\rangle - |\Phi^-\rangle)/\sqrt{2}$, has support on only one Bell subspace. Further, $|\Phi^0\rangle$ has only one ebit [19] of entanglement—yet even so, we can use this state to perform unit fidelity teleportation over an arbitrary distance.

Spin-1 models.—The Affleck-Kennedy-Lieb-Tasaki (AKLT) model [20] can be related to the antiferromagnetic Heisenberg model for spin 1, and is given by $\alpha = 1/3$ in the class of Hamiltonians

$$H_{\mathrm{AKLT}} = \sum_{i=1}^{N} \hat{S}_i.\hat{S}_{i+1} + \alpha(\hat{S}_i.\hat{S}_{i+1})^2. \tag{9}$$

We may decompose each spin-1 site, i, as two virtual spin-1/2 sites, $i, \bar{i}$, and project onto the spin-1 subspace. For an N site chain with spin-1/2 boundary conditions, we may write the ground state of the AKLT model as [21] $|\psi_{\mathrm{AKLT}}\rangle = (\otimes_{k=1}^{N} A_{k\bar{k}})|I\rangle$ where $|I\rangle = \otimes_{k=0}^{N}|I_{\bar{k}k+1}\rangle$ is a product of singlets, and $A_{k\bar{k}}$ projects the spins at sites k and $\bar{k}$ onto their symmetric subspace. The operation of projecting out the singlet components leaves the state in the Bell subspace which also contains the singlet quantum states.

Verstraete *et al.* [21] show that the projection onto Bell states in the $i \otimes \bar{i}$ space is achievable with only single particle measurements. Hence, one can use only single particle spin-1 measurement to teleport a spin-1/2 state along an AKLT chain. Moreover, a linear spin-1 Heisenberg antiferromagnetic may exhibit spin-1/2 degrees of freedom at the boundaries, as evidenced by both numerical [22] and experimental results [23]. Coupling the target state into the spin chain by a Bell-basis measurement over the target state and the boundary spin-1/2 degrees of freedom, the entire teleportation problem reduces to an initial Bell-basis measurement and single particle measurements on spin-1 particles [see also Ref. [24]].

Non-Bell-basis measurements.—Experimentally, it is very difficult to perform Bell-basis measurements directly, and we now consider teleportation using only single particle measurements. To affect a Bell-basis measurement, we may use the fact that there is a similarity transformation between Bell-basis projection operators and a product of single particle projections, by using an entangling operation, U[25]. A Bell projector P^j may be written as $P^j = U^\dagger P_2^{j_2} P_1^{j_1} U$ where j_1 and j_2 are functions of j set by the unitary operator U, and $P_i^{j_i}$ are single site projectors. Projections onto different sites commute, and hence, we may decompose our Bell measurement protocol as an application of entangling unitaries onto the whole quantum state, followed by a complete set of single particle projective measurements.

Generalizations.—Generalization to higher spin is possible, where a generalized N-Bell state is given by

$$|AB\} = \sum_{i=0}^{N-1} \omega^{Bi} |i(i + A)\rangle,$$

where $s = \frac{N-1}{2}$ is the spin, $\omega = \sqrt{[N]}1$ is a primitive Nth root of 1, and $A, B \in \{-s, \ldots, s\}$ and $|ij\rangle$ are the usual two-particle spin-s basis states. In analogy to the interpretation of σ_x as a bit flip and σ_z as a phase flip, we define the action of permutation and phase correction matrices to be:

$$P|l\rangle = |l + 1\rangle, \qquad Q|l\rangle = \omega^l |l\rangle.$$

As in the case of spin-1/2 particles, the corrections P^i and Q^i for $i = 1, 2, \ldots, N$ form a group, and we may perform exactly the same procedure of generalized Bell-basis measurement, followed by a cumulative correction at the end of the procedure.

We have shown that by using a Bell-basis measurement protocol, it is possible to decompose the Hilbert space of many spin-1/2 particles into several Bell subspaces, which may be understood as subspaces corresponding to distinct corrections. Further, we have shown that if any quantum state belongs to a Bell subspace, then we may perform unit fidelity teleportation with it, using only measurements in the Bell basis. We have presented a parameter whose magnitude provides a lower bound on the fidelity of teleportation. We have presented several model Hamiltonians for which the ground state is amenable to our teleportation protocol. Some alternative measurement schemes have been presented, including a scheme for a spin-1 system. Finally, we note that we have only considered the ability to teleport one spin quantum state to an arbitrary site [26] within a spin chain. Thus, v^0 and $v^0 \otimes v^0$ both have $\mathcal{O} = 3$, despite having a different number of ebits. There may be further insights to be gained into the entanglement content of a system by considering the ability to teleport several spin states to arbitrary sites.

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*Electronic address: jpb@physics.uq.edu.au

[1] J. Preskill, J. Mod. Opt. **47**, 127 (2000).
[2] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Phys. Rev. Lett. **78**, 2275 (1997).
[3] A. Wong and N. Christensen, Phys. Rev. A **63**, 044301 (2001).
[4] O. Biham, M. A. Nielsen, and T. J. Osborne, Phys. Rev. A **65**, 062312 (2002).
[5] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[6] $\sigma^x \otimes \sigma^x$ and $\sigma^z \otimes \sigma^z$ commute, and the Bell states are also eigenstates of $\sigma^y \otimes \sigma^y$ with $\sigma^y \otimes \sigma^y |kl\} = -lk|kl\}$.
[7] J. Links, J. P. Barjaktarevic, G. J. Milburn, and R. H. McKenzie, quant-ph/0506071.
[8] M. den Nijs and K. Rommelse, Phys. Rev. B **40**, 4709 (1989).
[9] M. A. Nielsen, Phys. Lett. A **308**, 96 (2003).
[10] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. **86**, 5188 (2001).
[11] A. M. Childs, D. W. Leung, and M. A. Nielsen, Phys. Rev. A **71**, 032318 (2005).
[12] M. P. A. Fisher, cond-mat/9806164.
[13] M. Popp, F. Verstraete, M. A. Martin-Delgado, and J. I. Cirac, Phys. Rev. A **71**, 042306 (2005).
[14] C. K. Majumdar and D. K. Ghosh, J. Math. Phys. (N.Y.) **10**, 1399 (1969).
[15] More formally the spaces, $V_{[jk]}^L$, are invariant under an action of the symmetric group.
[16] J. Chayes, L. Chayes, and S. Kivelson, Commun. Math. Phys. **123**, 53 (1989).
[17] Z. G. Soos and S. Ramasesha, Phys. Rev. B **29**, 5410 (1984).
[18] E. Ising, Z. Phys. **31**, 253 (1925).
[19] 1 ebit is equivalent to one distillable singlet.
[20] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Phys. Rev. Lett. **59**, 799 (1987).
[21] F. Verstraete, M. A. Martín-Delgado, and J. I. Cirac, Phys. Rev. Lett. **92**, 087201 (2004).
[22] S. R. White, Phys. Rev. Lett. **69**, 2863 (1992).
[23] M. Hagiwara, K. Katsumata, I. Affleck, B. I. Halperin, and J. P. Renard, Phys. Rev. Lett. **65**, 3181 (1990).
[24] H. Fan, V. Korepin, and V. Roychowdhury, Phys. Rev. Lett. **93**, 227203 (2004).
[25] C. H. Bennett, J. I. Cirac, M. S. Leifer, D. W. Leung, N. Linden, S. Popescu, and G. Vidal, Phys. Rev. A **66**, 012305 (2002).
[26] We have not assumed that our indices correspond to any physical ordering.