Theory of Decoherence due to Scattering Events and Lévy Processes

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A general connection between the characteristic function of a Lévy process and loss of coherence of the statistical operator describing the center of mass degrees of freedom of a quantum system interacting through momentum transfer events with an environment is established. The relationship with microphysical models and recent experiments is considered, focusing on the recently observed transition between a dynamics described by a compound Poisson process and a Gaussian process.

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The study of decoherence [1], both at theoretical and experimental level, owes its relevance to a twofold motivation: on the one and historically older hand it provides a fruitful research area for the exploration of the quantum classical boundary, on the other hand it is the formidable quantum enemy to be overcome or outwitted in order to actually realize quantum computers. Recently various experiments have been performed in which both qualitative and quantitative analyses of decoherence are feasible. These achievements both force and invite us to go beyond an implicitly established "common lore" [2], which sometimes deceitfully lets features of simplified models appear universal, contrary to experimental evidence [3]. In the present Letter we will focus on the issue of decoherence of the center of mass degrees of freedom of massive test particles, an object of recent and very accurate quantitative experimental investigations [4–7], showing how these different situations can be addressed within a unified theoretical approach which, exploring the most often fruitful connections between quantum and classical probability [8], puts into evidence how the loss of coherence in the off-diagonal position matrix elements of the statistical operator is generally described by the characteristic function (CF) of a Lévy process (LP). The common feature of the abovementioned experiments is the fact that, provided dissipative effects which take place on a much longer time scale are neglected, the interaction causing decoherence can be characterized through momentum transfer events, which following [9] we will generally call collisions; their effect can be described by means of a decoherence superoperator, a completely positive operation whose matrix elements in the position representation put into evidence a quantity often called decoherence function. In the Markovian case a common description of such dynamics can be obtained referring to the general structure of translation-covariant quantum-dynamical semigroups obtained by Holevo [10], relying on a quantum generalization of the Lévy-Khintchine formula. LP are a class of processes, including Gaussian processes, which despite obeying the Chapman-Kolmogorov equation characterizing Markov processes not necessarily have finite variance, so that the central limit theorem does not always apply. Such processes were in fact found looking for generalizations of such theorem, and are both space and time homogeneous, thus naturally arising when considering space translation invariance. The general structure of the CF, i.e., the Fourier transform of the probability density (PD), of such processes is given by the famous Lévy-Khintchine formula [for a most compact presentation see [11] and references therein]. The relevance of LP in physics is growing [12], since they allow to cope with situations not encompassed by the central limit theorem. This is therefore a natural way to improve the usual, almost ubiquitous models relying on linear coupling and Gaussian statistics, whose limitations in the description of open systems and, in particular, of decoherence begin to be appreciated [2,3,13].

We first start by introducing in a way adapted to our purposes the results by Holevo [10], later connecting them to microphysical derivations and experimental realizations. If the dynamics causing decoherence is Markovian and described in terms of momentum transfers, so that in the absence of an external potential one has translation invariance, the generator of the quantum-dynamical semigroup generally has the structure $d\hat{\rho}/dt = \mathcal{L}_G[\hat{\rho}] + \mathcal{L}_P[\hat{\rho}]$ with $\hat{\rho}$ the statistical operator of the test particle; $\mathcal{L}_G$ a so-called Gaussian component given by

$$\mathcal{L}_G[\hat{\rho}] = -ia[\hat{x}, \hat{\rho}] - \tfrac{1}{2}D[\hat{x}, [\hat{x}, \hat{\rho}]], \tag{1}$$

written for simplicity in the one-dimensional case, with $a \in \mathbb{R}$, $D > 0$, and $\hat{x}$ the position operator of the test particle; $\mathcal{L}_P$ the so-called Poisson component

$$\begin{aligned}\mathcal{L}_P[\hat{\rho}] = &\int dq |\lambda(q)|^2 [e^{(i/\hbar)q\hat{x}}\hat{\rho}e^{-(i/\hbar)q\hat{x}} - \hat{\rho}] \\ &+ 2\int dq \Re(\omega(q)\lambda^*(q))[e^{(i/\hbar)q\hat{x}}\hat{\rho}e^{-(i/\hbar)q\hat{x}} - \hat{\rho}] \\ &+ \int dq |\omega(q)|^2 \bigg[e^{(i/\hbar)q\hat{x}}\hat{\rho}e^{-(i/q)q\hat{x}} - \hat{\rho} \\ &- \frac{i}{\hbar}\frac{q[\hat{x}, \hat{\rho}]}{1 + q^2/q_0^2}\bigg],\end{aligned} \tag{2}$$

where $|\omega(q)|^2 dq$ is a positive measure, also called Lévy measure, with $|\omega(q)|^2$ possibly divergent in zero but such that the Lévy condition $\int dq |\omega(q)|^2 q^2/(1 + q^2) < \infty$

holds, the weights $\lambda(q)$ and $\omega(q)$ are in the general case complex functions, the integration variable q has the dimension of momentum and the meaning of momentum transfer, the parameter q_0 only appearing for dimensional purposes in the regularizing factor. In stating the result we have neglected free evolution and dissipative effects which are relevant only on a much longer time scale, so that the momentum of the test particle has essentially been treated as a $\mathbb{C}$ number. Focusing on the position matrix elements the master equation takes the form

$$\frac{d}{dt}\langle x|\hat{\rho}|y\rangle = -\Psi(x-y)\langle x|\hat{\rho}|y\rangle \tag{3}$$

with $\Psi(x-y)$ given by (note the dependence on $x-y$ according to translation invariance)

$$\begin{aligned}\Psi(x-y) = {}& ia(x-y) + \frac{1}{2}D(x-y)^2 \\ & - \int dq|\lambda(q)|^2[e^{(i/\hbar)q(x-y)} - 1] \\ & - 2\int dq \Re(\omega(q)\lambda^*(q))[e^{(i/\hbar)q(x-y)} - 1] \\ & - \int dq|\omega(q)|^2\left[e^{(i/\hbar)q(x-y)} - 1 - \frac{i}{\hbar}\frac{q(x-y)}{1+q^2/q_0^2}\right],\end{aligned} \tag{4}$$

so that one immediately has the general solution

$$\langle x|\hat{\rho}_t|y\rangle = e^{-t\Psi(x-y)}\langle x|\hat{\rho}_0|y\rangle. \tag{5}$$

The function $\Phi(t, x-y) \equiv e^{-t\Psi(x-y)}$ is the CF of a LP, $\Psi(x-y)$ being called its characteristic exponent, the quantity actually fully characterized by the Lévy-Khintchine formula. The fact that $\Phi(t, x-y)$ is a CF automatically entails that its modulus is less than one and the value one for $x-y$ tending to zero, i.e., the natural properties in order to predict the reduction of the off-diagonal matrix elements in (5) due to decoherence. This suppression of coherence, however, happens with a variety of behaviors going far beyond the quadratic common lore corresponding to Gaussian statistics, depending on the process characterizing the physical interaction.

We now briefly present some microphysical models giving specific realizations of (3) and make later contact with actual experiments; as it turns out Eq. (3) actually encompasses all known models of decoherence for the center of mass degrees of freedom [1]. Let us first consider the motion of a massive test particle interacting through collisions with a background gas, developed in detail in [14], where also dissipative effects have been taken into account, relying on a kinetic approach. Neglecting free motion and dissipation the result becomes

$$\begin{aligned}\frac{d}{dt}\langle \boldsymbol{x}|\hat{\rho}|\boldsymbol{y}\rangle = {}& n(2\pi)^4\hbar^2\int d^3\boldsymbol{q}|\tilde{t}(q)|^2 S(\boldsymbol{q}, E) \\ & \times [e^{(i/\hbar)\boldsymbol{q}\cdot(\boldsymbol{x}-\boldsymbol{y})} - 1]\langle \boldsymbol{x}|\hat{\rho}|\boldsymbol{y}\rangle,\end{aligned} \tag{6}$$

where n is the gas density, $\tilde{t}(q)$ the Fourier transform of the interaction potential, and S a two-point correlation function characterizing the gas known as dynamic structure factor depending on both momentum and energy transfer ($\boldsymbol{q}$ and E). For a finite macroscopic scattering cross section $\sigma = (2\pi)^4\hbar^2(M/p_0)\int d^3\boldsymbol{q}|\tilde{t}(q)|^2 S(\boldsymbol{q}, E)$, with M the mass of the test particle and p_0 its incoming momentum, one can introduce a scattering rate $\Lambda \equiv n(p_0/M)\sigma$ and a suitably normalized PD

$$\mathcal{P}(\boldsymbol{q}) = \frac{n}{\Lambda}(2\pi)^4\hbar^2|\tilde{t}(q)|^2 S(\boldsymbol{q}, E), \tag{7}$$

so that (5) reads

$$\langle \boldsymbol{x}|\hat{\rho}_t|\boldsymbol{y}\rangle = e^{-\Lambda[1-\Phi_{\mathcal{P}}(\boldsymbol{x}-\boldsymbol{y})]t}\langle \boldsymbol{x}|\hat{\rho}_0|\boldsymbol{y}\rangle, \tag{8}$$

where we have introduced the CF $\Phi_{\mathcal{P}}$ associated to the PD $\mathcal{P}$, i.e., its Fourier transform. Here no confusion should arise: the exponential function in (8) is the CF of a LP which in this particular case can be expressed in terms of the CF $\Phi_{\mathcal{P}}$ of the PD $\mathcal{P}$. Equation (8) is a particular realization of (5) given by the choice $a = D = 0$, $\omega(q) = 0$, and $|\lambda(q)|^2 \to \Lambda\mathcal{P}(\boldsymbol{q})$ in (4), corresponding to a compound Poisson process [15]. The physical picture behind it is the following: the dynamics is driven by collisions, the probability of having a definite number of collisions in a time t being given by a Poisson distribution with intensity Λ and mean Λt; each collision, however, is not characterized by a fixed, deterministic value of the transferred momentum $\boldsymbol{q}$, but rather by a certain PD $\mathcal{P}(\boldsymbol{q})$ depending in the case under consideration on the two-body interaction potential and a suitable correlation function. Leaving aside for a moment the detailed structure of (6) related to its microphysical derivation, the result (8) generally applies to a situation in which one has a collection of momentum transfer events each characterized by a certain PD (to be obtained or introduced by means of some microscopic or phenomenological model) corresponding to a compound Poisson process. Note that the fact that the probability of having a certain number of events is Poisson distributed is crucial in order to have a Markovian dynamics [16], as we shall see later on. The result (8) embraces the work by Gallis and Fleming [17], which apart from a simple but relevant correction [18] has been used for the theoretical analysis of decoherence experiments with fullerenes, both in the case of collisional decoherence [4] and of decoherence due to thermal emission of radiation [5]. Both situations correspond to compound Poisson processes, where the relevant PD $\mathcal{P}(\boldsymbol{q})$ is obtained in terms of the collisional cross section and the spectral photon emission rate, respectively [19]. According to a detailed theoretical analysis [20] the final visibility is obtained by an average of the characteristic exponent in (8) over the possible scattering positions in the interferometer. Furthermore, in the case of collisional decoherence the random momentum kicks are so strong that the CF $\Phi_{\mathcal{P}}$ in (8) is essentially zero for the path separations of interest, so that its actual structure is not relevant and only the mean Λt determines the fringes visibility. The connection of Eq. (8) with the common lore of a Gaussian process is straightforward [14]; expanding the exponential in (6) up to second order, the solution rather than (8) becomes

$$\langle \boldsymbol{x}|\hat{\rho}_t|\boldsymbol{y}\rangle = e^{-\Lambda[-i\langle \boldsymbol{q}\rangle\cdot(\boldsymbol{x}-\boldsymbol{y})+1/2\sum\langle q_i q_j\rangle(x_i-y_i)(x_j-y_j)]t}\langle \boldsymbol{x}|\hat{\rho}_0|\boldsymbol{y}\rangle, \tag{9}$$

where $\langle \boldsymbol{q}\rangle \equiv \int d^3\boldsymbol{q}\mathcal{P}(\boldsymbol{q})\boldsymbol{q}$ and $\langle q_i q_j\rangle \equiv \int d^3\boldsymbol{q}\mathcal{P}(\boldsymbol{q})q_i q_j$ are the moments of the PD $\mathcal{P}$ appearing by definition (if they exist) as coefficients in the Taylor expansion of the CF $\Phi_{\mathcal{P}}$. One thus ends up with the CF of a Gaussian process with mean given by the product of the intensity Λ and the first moment of the distribution $\mathcal{P}$ characterizing the original compound Poisson process, and variance given by the product of intensity and second moments, corresponding to the choice $a \to -\Lambda\langle \boldsymbol{q}\rangle$ and $D \to D_{ij} = \Lambda\langle q_i q_j\rangle$ in (4), $\lambda(q)$ and $\omega(q)$ being zero. As a last example we consider the case of a massive test particle interacting with a chaotic environment, modeled through random matrices. In the absence of an external potential and considering an environment with constant average level density the dynamics is given by [21]

$$\frac{d}{dt}\langle x|\hat{\rho}|y\rangle = K\left[G\left(\frac{x-y}{x_0}\right) - 1\right]\langle x|\hat{\rho}|y\rangle, \tag{10}$$

where G is directly related to a two-point correlation function describing the chaotic background, with characteristic correlation length x_0, while K is a coupling constant. In the weak-coupling limit the authors of [21] propose the expression $G(r) \approx 1 - |r|^\alpha$ requiring $\alpha \in (0, 2]$ due to some necessary restriction on the two-point correlation function, so that (10) has the simple solution

$$\langle x|\hat{\rho}_t|y\rangle = e^{-K|(x-y)/x_0|^\alpha t}\langle x|\hat{\rho}_0|y\rangle. \tag{11}$$

They then show that in this case the statistical operator can display dynamics given by so-called Lévy stable laws. This is a naturally expected result in the present framework since (11) is a particular case of (5) corresponding to $a = D = 0$, $\lambda(q) = 0$ and a Lévy measure $|\omega(q)|^2 \propto 1/|q|^{\alpha+1}$ corresponding to the symmetric stable LP with scaling exponent α [15]; the restriction on α now arises due to the Lévy condition, the case $\alpha = 2$ corresponding to a Gaussian process, all other symmetric Lévy stable laws having infinite second moments. Suppression of spatial coherence for random momentum transfers governed by a Lévy stable law is expected to be stronger than for the usual Gaussian case [22], even though no experimental evidence is available yet.

We now consider the transition between (8) and (9), i.e., from the CF of a compound Poisson process to that of the related Gaussian process, in view of recent experiments on decoherence in an atom interferometer, obtained by spontaneous scattering of photons off atoms interacting in a controlled way with a laser [7]. We will focus, in particular, on the most recent results [6] in which both single- and multiple-photon decoherence has been observed, noting that "the few-photon limit is of a qualitatively different character" and following "the smooth transition between these two regimes", connecting the many-photon limit with the "common lore" master equation [3] predicting exponential reduction in coherence with separation squared. For the case of an atom interacting with a laser, the PD that the atom experiences a given momentum transfer along the direction of propagation of the laser as a consequence of spontaneous emission has been characterized by Mandel [23]; let us call it $\mathcal{P}_M(q)$ for the case of a single photon. We can now therefore write the master equation for the case at hand in analogy to (6) in operator form as follows

$$\frac{d}{dt}\hat{\rho} = \Gamma\int dq\mathcal{P}_M(q)[e^{(i/\hbar)q\hat{x}}\hat{\rho}e^{-(i/\hbar)q\hat{x}} - \hat{\rho}], \tag{12}$$

where Γ is once again a scattering rate depending, e.g., on the intensity of the laser. In order to make contact with the analysis put forward in [6] we formally write the solution as a Dyson expansion, thus describing the time evolution as a sequence of jumps, given by the random momentum transfers described by $\mathcal{P}_M(q)$, on the background of a relaxing evolution, trivial for the case at hand in which we neglect free dynamics and dissipation. The jump expansion reads

$$\begin{aligned}\hat{\rho}_t &= e^{-\Gamma t}\hat{\rho}_0 + \sum_{n=1}^{\infty}\int_0^t dt_n\int_0^{t_n}dt_{n-1}\ldots\int_0^{t_2}dt_1 e^{-\Gamma(t-t_n)}\\ &\quad\times\Gamma\mathcal{J}_{\mathcal{P}_M}e^{-\Gamma(t_n-t_{n-1})}\ldots e^{-\Gamma(t_2-t_1)}\Gamma\mathcal{J}_{\mathcal{P}_M}e^{-\Gamma t_1}\hat{\rho}_0\\ &= \sum_{n=0}^{\infty}\frac{(\Gamma t)^n}{n!}e^{-\Gamma t}\underbrace{\mathcal{J}_{\mathcal{P}_M}\circ\ldots\circ\mathcal{J}_{\mathcal{P}_M}}_{n\text{ times}}[\hat{\rho}_0],\end{aligned} \tag{13}$$

with $\mathcal{J}_{\mathcal{P}_M}$ a decoherence superoperator given by the following completely positive, trace preserving operation

$$\mathcal{J}_{\mathcal{P}_M}[\hat{\rho}] \equiv \int dq\mathcal{P}_M(q)e^{(i/\hbar)q\hat{x}}\hat{\rho}e^{-(i/\hbar)q\hat{x}}, \tag{14}$$

where $\mathcal{P}_M$ is a PD and the $e^{(i/\hbar)q\hat{x}}$ are momentum translation operators. This decoherence superoperator generally describes the effect on the statistical operator of a momentum transfer randomly distributed according to $\mathcal{P}_M$. The matrix elements of the decoherence superoperator in the position representation give a function often called decoherence function [4,6], actually the CF associated to $\mathcal{P}_M$, with all its natural properties, including the fact that it is positive definite, corresponding to the complete positivity of the decoherence superoperator $\mathcal{J}_{\mathcal{P}_M}$. The Mandel PD $\mathcal{P}_M$ leads to

$$\begin{aligned}\langle x|\mathcal{J}_{\mathcal{P}_M}[\hat{\rho}]|y\rangle &= \Phi_{\mathcal{P}_M}(x-y)\langle x|\hat{\rho}|y\rangle\\ &= \frac{3}{2}e^{ik_0(x-y)}\Big\{\mathrm{sinc}[k_0(x-y)]\\ &\quad+\frac{\cos[k_0(x-y)]-\mathrm{sinc}[k_0(x-y)]}{[k_0(x-y)]^2}\Big\}\\ &\quad\times\langle x|\hat{\rho}|y\rangle,\end{aligned} \tag{15}$$

with k_0 the wave vector of the exciting light, and using

$$\underbrace{J_{\mathcal{P}_M} \circ \ldots \circ J_{\mathcal{P}_M}}_{n\,\text{times}}[\hat{\rho}] = \int dq (\underbrace{\mathcal{P}_M * \ldots * \mathcal{P}_M}_{n\,\text{times}})(q) \times e^{(i/\hbar)q\hat{x}} \hat{\rho} e^{-(i/\hbar)q\hat{x}},$$

with $\circ$ the composition of superoperators and $*$ the convolution of PD [the convolution n times of $\mathcal{P}_M$ giving according to [23] the PD that a momentum transfer q is imparted to the atom as a consequence of n spontaneous emissions], the matrix elements of Eq. (13) become

$$\langle x|\hat{\rho}_t|y\rangle = \sum_{n=0}^{\infty} \frac{(\Gamma t)^n}{n!} e^{-\Gamma t} \Phi^n_{\mathcal{P}_M}(x-y)\langle x|\hat{\rho}_0|y\rangle \equiv \sum_{n=0}^{\infty} p_n(t) \Phi^n_{\mathcal{P}_M}(x-y)\langle x|\hat{\rho}_0|y\rangle, \tag{16}$$

where according to the property of the Fourier transform the nth power of the CF $\Phi_{\mathcal{P}_M}$ appears; note that if the scattering rate is assumed time dependent, according to a time inhomogeneous Poisson process, nothing would change but the replacement $\Gamma \rightarrow \int_0^t dt' \Gamma(t')$. If the $p_n(t)$ are Poisson distributed with mean $\bar{n} \equiv \Gamma t$, where t is the time of interaction with the laser, Eq. (16) is exactly equivalent to Eq. (8) and this is the only distribution of the weights $p_n(t)$ describing a Markovian dynamics [16]. For the decoherence experiments in atom interferometry [6] the relative contrast is directly related to the modulus of the CF in (5), so that provided the dynamics is Markovian switching from the single- to the many-photon limit for growing intensity of the laser the compound Poisson process characterized by $\Phi_{\mathcal{P}_M}$, and described by (8) or (16), goes over to the related Gaussian process described by (9). This is essentially what has been observed for the first time in [6]: the smooth transition between the two qualitatively distinct regimes can therefore be understood and described in a unified way on the basis of the presented theoretical framework, expressing the loss of spatial coherence in terms of the CF of a suitable LP. In particular, the authors of [6] compare their results with the master equation only in the many-photon limit, when the "common lore" quadratic expression applies, apart from the correction due to the nonvanishing first moment reflecting anisotropy; in the single- or few-photon limit they rely on a formula like the jump expansion (16) of the master equation, fitting from the very beginning the experiment with a Gaussian distribution for $p_n(t)$ (though possibly allowing for a Poisson relationship between mean and variance), rather than with a Poisson distribution corresponding to the ideal case [24] which describes a Markovian dynamics. These small corrections notwithstanding, depending on deviations of the atom laser interaction from the Markov regime, these experiments have obtained the first experimental study of the transition between the decoherence regimes described by Eqs. (8) and (9), respectively.

A general theoretical description of decoherence due to random momentum transfers has been presented, showing how spatial coherence is suppressed according to the CF of a LP. This has been obtained relying on the general structure of translation-covariant generators of quantum-dynamical semigroups derived by Holevo as a quantum Lévy-Khintchine formula. Different microscopic models have been shown to lead to particular examples of the general structure, not only Gaussian processes, but also compound Poisson and symmetric stable LP have been considered, thus going beyond the usual limitation given by Gaussian statistics and opening the way for both microscopical and phenomenological treatment of new scenarios, especially in connection with chaotic environments.

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