Identifying Turbulent Energy Distributions in Real, Rather than Fourier, Space

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It has been suggested that the equilibrium-range properties of high-Reynolds number turbulence are more readily observed in spectral space, using $E(k)$ or $T(k)$, than in real space, using second- or third-order structure functions. For example, the $-5/3$ law is usually easier to see in experimental data than the equivalent $2/3$ law. We argue that this is not an implicit feature of a real-space description of turbulence. Rather, it is because the second-order structure function mixes small and large-scale information. To remedy this problem we adopt a real-space function, the signature function, which plays the role of an energy density, somewhat analogous to $E(k)$. In this Letter we determine the form of the signature function in a variety of turbulent flows. We find that dissipation-range phenomena, such as the so-called bottleneck effect, are evident in the signature function, while absent in the structure function.

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Since the pioneering work of Richardson and Kolmogorov, one of the central challenges in turbulence has been the need to quantify the variation of energy with eddy size. Traditionally there have been two means of describing how kinetic energy is distributed amongst the hierarchy of eddy sizes found in isotropic turbulence. These are the three-dimensional energy spectrum, $E(k)$, and the second-order structure function, $\langle[\Delta \mathrm{v}]^2\rangle(r) = \langle[u_x(\mathbf{x} + r\hat{\mathbf{e}}_x) - u_x(\mathbf{x})]^2\rangle$. [We shall use the notation set out in [1] throughout.] However, it is well known that both of these methods are, to some degree, problematic. Here we explore a third approach, recently advocated in [1], and provide the first assessment of this approach using laboratory data.

Let us start by summarizing the strengths and weaknesses of the energy spectrum and structure function as a means of describing energy distributions. The utility of the energy spectrum rests, in part, on three useful properties of $E(k)$: (i) it is positive; (ii) it integrates to give the kinetic energy, $\int E(k)dk = \frac{1}{2}\langle\mathbf{u}^2\rangle$; and (iii) a random distribution of eddies (blobs of vorticity) of fixed size ℓ_e produces an energy spectrum of the form $E(k) \sim \langle\mathbf{u}^2\rangle\ell_e(k\ell_e)^4 \times \exp[-(k\ell_e)^2/4]$, which exhibits a fairly sharp peak at $k \sim \pi/\ell_e$. [See, for example, [1,2].] The first two properties suggest that we may regard $E(k)$ as an energy distribution in spectral space, while the third indicates that we may loosely associate eddy size with π/k. However, it is important to note that eddies of a given size produce a distributed energy spectrum, albeit peaked around π/ℓ_e. Thus we must be cautious in making the connection between eddy size and wave number, particularly in the range $k < \pi/\ell$ and $k > \pi/\eta$ where, as shown in [1,2], $E(k)$ has nothing at all to do with the energy of eddies of size π/k. (Here ℓ and η are the integral and Kolmogorov scales, respectively.)

The second-order structure function also has problems, as we now show. A common but naïve interpretation of $\langle[\Delta \mathrm{v}]^2\rangle(r)$ is that it represents the energy of eddies of size r. The first hint that this is a flawed interpretation comes from extending the idea to the longitudinal vorticity structure function $\langle[\Delta\omega]^2\rangle(r)$, which, according to this viewpoint, should be indicative of the enstrophy of eddies of size r. However, in the inertial range the $2/3$ law yields $\frac{3}{2}\langle[\Delta\omega]^2\rangle = \langle\boldsymbol{\omega}^2\rangle[1 - (11/3)\beta(r/\eta)^{-4/3}]$, where β is the Kolmogorov constant and η the Kolmogorov length. Clearly the vorticity structure function is dominated by the dissipation-range vorticity, contrary to the above viewpoint.

A slightly more sophisticated but still naïve interpretation of $\langle[\Delta \mathrm{v}]^2\rangle(r)$ is to consider it as a form of cumulative energy density. The rationale goes as follows. Eddies of size much less than the separation r can induce a large signal at $\mathbf{x}$ or $\mathbf{x}' = \mathbf{x} + r\hat{\mathbf{e}}_x$, but not at both points simultaneously. Thus eddies smaller than r tend to induce a contribution to $[\Delta \mathrm{v}]^2$ which is of the order of their kinetic energy. On the other hand, eddies much greater than r tend to produce similar velocities at both $\mathbf{x}$ and $\mathbf{x}'$, and so make little contribution to the velocity difference, $\Delta \mathrm{v}$. So we might think of the structure function as a sort of filter, suppressing information from eddies of size greater than r. Given that $\frac{3}{4}\langle[\Delta \mathrm{v}]^2\rangle \rightarrow \frac{1}{2}\langle\mathbf{u}^2\rangle$ for large r we might expect that

$$\frac{3}{4}\langle[\Delta \mathrm{v}]^2\rangle(r) \sim [\text{energy in eddies of size } < r] \sim \int_{\pi/r}^{\infty} E(k)dk, \tag{1}$$

and indeed such estimates are commonly made [2,3]. This led Townsend [2] to suggest that

$$V_T(r) = d\langle\tfrac{3}{4}[\Delta \mathrm{v}]^2\rangle/dr \tag{2}$$

acts as a kind of energy density, playing a role analogous to $E(k)$. However, this view is also flawed. Eddies whose sizes are much larger than r make a contribution to $\frac{3}{4}[\Delta \mathrm{v}]^2$ of the order of $\frac{3}{4}\langle(\partial u_x^L/\partial x)^2\rangle r^2 = \frac{1}{10}\langle\frac{1}{2}(\boldsymbol{\omega}^L)^2\rangle r^2$, where the superscript L indicates a contribution from eddies of size much larger than r [1]. So we should replace (1) by the estimate

$$\frac{3}{4}\langle[\Delta \mathbf{v}]^2\rangle \sim [\text{energy in eddies of size } < r] + (r^2/10)[\text{enstrophy in eddies of size } > r]. \tag{3}$$

Indeed it is readily confirmed, using the transform pair which relates $E(k)$ to $\langle[\Delta \mathrm{v}]^2\rangle(r)$, that a good approximation to the relationship between these two quantities is

$$\frac{3}{4}\langle[\Delta \mathrm{v}]^2\rangle(r) \approx \int_{\pi/r}^{\infty} E(k)dk + (r/\pi)^2 \int_0^{\pi/r} k^2 E(k)dk, \tag{4}$$

which is more or less what we would expect from (3). [See [1] for a more detailed discussion.] Thus the structure function mixes large- and small-scale information, as well as information about energy and enstrophy. It follows that (2) is not a satisfactory estimate of the kinetic energy density. This becomes particularly problematic in two-dimensional turbulence where, in the inertial range, $E \sim k^{-n}$, $n > 3$. In such cases the second integral on the right of (4) dominates $\langle[\Delta \mathrm{v}]^2\rangle$, and so $\langle[\Delta \mathrm{v}]^2\rangle \sim r^2$ whatever the exponent n, provided that n exceeds 3 [1,4].

This unsatisfactory state of affairs led Davidson [1] to introduce a new quantity, called the *signature function*, which seeks to eliminate the large-scale information contained in (3) and (4). Like $\langle[\Delta \mathrm{v}]^2\rangle$, it is a real-space function and so does not rely on a Fourier decomposition of the velocity field.

The signature function is defined for isotropic turbulence as follows:

$$V(r) = -\frac{1}{2} r^2 \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{3}{4}\langle[\Delta \mathrm{v}]^2\rangle\right], \tag{5}$$

$$\frac{3}{4}\langle[\Delta \mathrm{v}]^2\rangle(r) = \int_0^r V(s)ds + r^2 \int_r^{\infty} V(s)/s^2 ds. \tag{6}$$

It is readily confirmed that V has the properties: (i) $\int_0^r V(r)dr \geq 0$; (ii) $\int_0^{\infty} V(r)dr = \frac{1}{2}\langle \mathbf{u}^2\rangle$; and (iii) a random distribution of eddies (spatially compact vortex blobs) of fixed size ℓ_e gives rise to the signature function $V(r) \sim \langle \mathbf{u}^2\rangle \ell_e^{-1}(r/\ell_e)^3 \exp[-r^2/\ell_e^2]$, which has a sharp peak around $r \sim \ell_e$. [See [1] for more details.] If we compare these properties with those of $E(k)$ we see that, like the energy spectrum, $V(r)$ may be thought of as an energy density, with r interpreted as eddy size. One way to understand the rationale behind definition (5) and (6) is to integrate (5) and use isotropy to write the result in the form

$$\int_0^r V(s)ds = \frac{3}{4}\langle 2[\Delta \mathrm{v}]^2 - [\Delta \mathrm{v}]_\perp^2\rangle. \tag{7}$$

Here $\langle[\Delta \mathrm{v}]_\perp^2\rangle = \langle[u_y(\mathbf{x} + r\hat{\mathbf{e}}_x) - u_y(\mathbf{x})]^2\rangle$ is the transverse structure function. Let us now consider the turbulence to be composed of an ensemble of eddies of size s_1, plus an ensemble of size s_2 etc. from η up to ℓ. Each ensemble is taken to be isotropic in its own right, as discussed in Davidson [1], pp. 418–419. When $s \ll r$ we refer to the corresponding ensemble as composed of small eddies, and when $s \gg r$ we refer to the eddies as large, using superscripts Sm and L to indicate small and large. Now an ensemble of eddies whose size is much less than r has a correlation length much smaller than r. Consequently, such an ensemble will make a contribution to the longitudinal and transverse structure functions of $\langle[\Delta \mathrm{v}^{Sm}]^2\rangle = 2\langle(u_x^{Sm})^2\rangle$ and $\langle[\Delta \mathrm{v}^{Sm}]_\perp^2\rangle = 2\langle(u_y^{Sm})^2\rangle$ which, by virtue of isotropy, are equal. So such eddies make a contribution to the right-hand side of (7) which is of the order of their kinetic energy, $\frac{1}{2}\langle(\mathbf{u}^{Sm})^2\rangle$. Eddies whose size is much greater than r, on the other hand, make a contribution to the right of (7) of $\frac{3}{4}\langle 2[\partial u_x^L/\partial x]^2 - [\partial u_y^L/\partial x]^2\rangle r^2$, which, by virtue of the isotropy of these eddies, is zero [1]. Thus definition (5) is an improvement over Townsend's function (2) to the extent that it filters out, at least partially, the large-scale information represented by the second integral on the right of (4).

Note that the formal relationship between $E(k)$ and $V(r)$ is readily shown to take the form of a Hankel transform pair,

$$E(k) = \frac{2\sqrt{2}}{3\sqrt{\pi}} \int_0^{\infty} rV(r)(rk)^{1/2} J_{7/2}(rk)dr \tag{8}$$

$$rV(r) = \frac{3\sqrt{\pi}}{2\sqrt{2}} \int_0^{\infty} E(k)(rk)^{1/2} J_{7/2}(rk)dk \tag{9}$$

from which it may be shown that

$$rV(r) \approx [kE(k)]_{k=\hat{\pi}/r}, \qquad \eta < r < \ell, \tag{10}$$

where $\hat{\pi} = 9\pi/8$ [1]. For example, the difference between $rV(r)$ and $[kE(k)]_{k=\hat{\pi}/r}$ can be shown to be less than 4% for power-law spectra of the form, $E = Ak^n$, $-2 < n < 1$. One illustration of this is the 2/3 law $\langle[\Delta \mathrm{v}]^2\rangle(r) = \beta\varepsilon^{2/3}r^{2/3}$, whose spectral equivalent is $E(k) = 0.761\beta\varepsilon^{2/3}k^{-5/3}$. In terms of $V(r)$ we have, from (5),

$$rV(r) = \tfrac{1}{3}\beta\varepsilon^{2/3}r^{2/3} = 1.016[kE(k)]_{k=\hat{\pi}/r}. \tag{11}$$

Note, however, that (10) does not apply for $r \ll \eta$ or $r \gg \ell$ since V and E have different limiting forms in these ranges [1].

The theoretical properties of the signature function are discussed in [1]. The primary purpose of this Letter is to investigate the ability of $V(r)$ to identify the equilibrium-range properties of laboratory turbulence, such as the inertial range scaling and the so-called bottleneck effect, which is an overshoot in the energy density at the dissipation end of the inertial range [5]. So let us turn to the experimental data.

The experimental data used for the present study are measurements made in fully developed grid and wake flows. The experimental conditions are described in [6] and need not be repeated in detail here. We note only the key points. The majority of the data relate to grid turbulence, in which the grid is located in a recirculatory wind tunnel of test section dimensions $1.8 \times 2.7\ \mathrm{m}^2$ and 11 m in length, while the grid itself consists of a perforated plate superimposed over a bi-plane grid of square rods. Measurements were made 40 mesh lengths downstream of the grid. Additional data relate to grid, plate, and circular cylinder wake flows, acquired in a blow-down wind tunnel of test section dimensions $35 \times 35\ \mathrm{cm}^2$ and 2 m in length, with measurements being made on the centerline of the wakes at a downstream location of $40d$. For all flows u_x is measured on the mean shear profile centerline using constant temperature hot-wire anemometry with a single-wire probe made of 1.27 μm diameter Wollaston (Pt-10% Rh) wire. The wire resolution is, at worst, $\sim 2\eta$ in the wake flows and $\sim 4\eta$ in the grid turbulence.

The plots of $V(r)$ were obtained from the 1D spectra by first converting them to structure functions, using the Fourier transform, and then determining V in accordance with (5). It was found necessary to smooth the structure function data prior to differentiation, and this was done by fitting a high-order polynomial to the data.

Figure 1 shows $3rV(r)$ and $\langle[\Delta \mathrm{v}]^2\rangle(r)$, normalized by $\varepsilon^{2/3}r^{2/3}$ and plotted against r/η, for grid turbulence at $R_\lambda = 255$. The data are from the high-Reynolds number grid experiment described in [6]. The vertical lines indicate the integral scale, the Taylor scale, and a value of 6η, which is often taken as an approximate minimum eddy size. According to (11) the inertial range in such compensated plots should show up as a plateau with a numerical value equal to the Kolmogorov constant, β. It is clear that, because of the modest value of Re, only a limited inertial range is discernible in the signature and structure functions. Nevertheless, the expected overshoot in energy at the junction of the inertial and dissipation ranges shows up clearly in the signature function, though not in the structure function. The cause of this overshoot, which has become known as the bottleneck effect [5], are the viscous forces [1]. Figure 2 shows compensated plots of $3rV(r)$ for grid turbulence ($R_\lambda = 255$) and for wakes behind a plate ($R_\lambda = 248$) and a cylinder ($R_\lambda = 254$). This time we use a linear plot and restrict ourselves to $r < l$, which corresponds to $r/\eta \sim 300$. (Of course, the large scales are anisotropic in the wake flows, so no significance should be attached to the corresponding plots of $V(r)$ in the vicinity of $r \sim l$. Note also that the differences in the curves at $r \sim l$ probably reflect a lack of universality at the large scales.) The inertial range shows up more clearly in these plots, with a Kolmogorov constant of around $\beta \approx 2.0$, which is in line with most estimates [7]. In Fig. 3 we show the effect of increasing R_λ on $3rV(r)$. As in Fig. 1, this corresponds to grid turbulence as described in [6], only this time we have $R_\lambda = 400 \rightarrow 1200$. Note that, for the highest value of R_λ shown, the signature function has acquired the characteristic double humped shape seen in energy spectra at large Re. The bump on the right is often interpreted as a consequence of inertial range intermittency [8], though it could also be caused by the time dependence of the large scales [1]. Finally, in Fig. 4, we the compare $rV(r)$ $[kE(k)]_{k=\hat{\pi}/r}$ for grid turbulence at $R_\lambda = 440$. It is clear that the two curves are very similar, as suggested by (10). We conclude by noting that these ideas may be extended to third-order statistics. Since $V(r)$ is the real-space analogue of $E(k)$, it is natural to seek the real-space counterpart of the transfer function $T(k)$. We start by integrating the Karman-Howarth equation. This yields an evolution equation for $V(r)$

$$\frac{\partial V}{\partial t} = \frac{\partial \Pi_V}{\partial r} + 2\nu\left[\frac{\partial}{\partial r}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V) - \frac{10}{r^2}V\right], \qquad (12)$$

$$\begin{aligned}\Pi_V &= \frac{1}{8}r^3\frac{\partial}{\partial r}r^{-6}\frac{\partial}{\partial r}[r^4\langle[\Delta \mathrm{v}]^3\rangle] \\ &= \frac{3}{4}\frac{\partial}{\partial r}\langle(\Delta \mathrm{v})_\perp^2(\Delta \mathrm{v})\rangle - \frac{3}{2}r^{-1}\langle(\Delta \mathrm{v})^3\rangle. \qquad (13)\end{aligned}$$

[See [1] for the details.] We might compare this with the equivalent spectral equation

$$\partial E/\partial t = -\partial \Pi_E/\partial k - 2\nu k^2 E,$$

where Π_E is the *spectral kinetic energy flux*. Evidently the function Π_V, like Π_E, captures the influence of the nonlinear inertial forces. If we are away from the dissipation

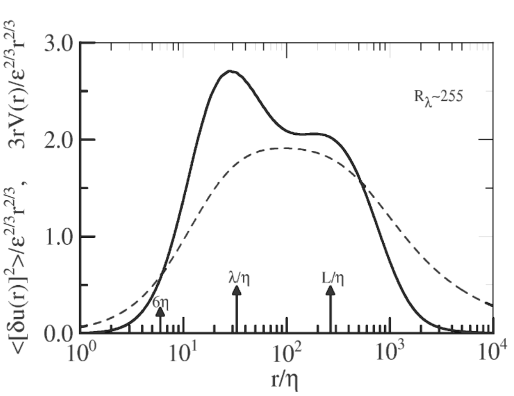

FIG. 1. Curves of $3rV(r)$ (solid line) and $\langle[\Delta \mathrm{v}]^2\rangle(r)$ (dashed line), normalized by $\varepsilon^{2/3}r^{2/3}$ and plotted against r/η, for grid turbulence at $R_\lambda = 255$.

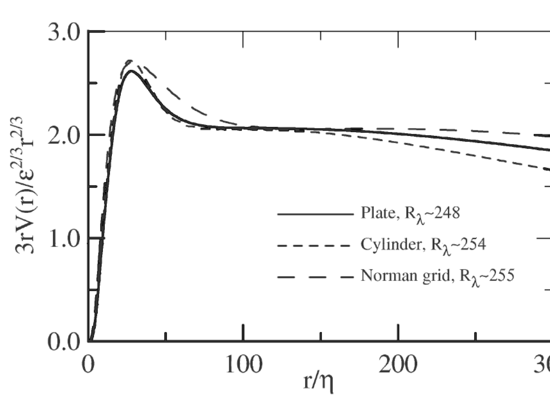

FIG. 2. Compensated plots of $3rV(r)$ for grid turbulence (upper dashed line, $R_\lambda = 255$) and wakes behind a plate (solid line, $R_\lambda = 248$) and a cylinder (lower dashed line, $R_\lambda = 254$).

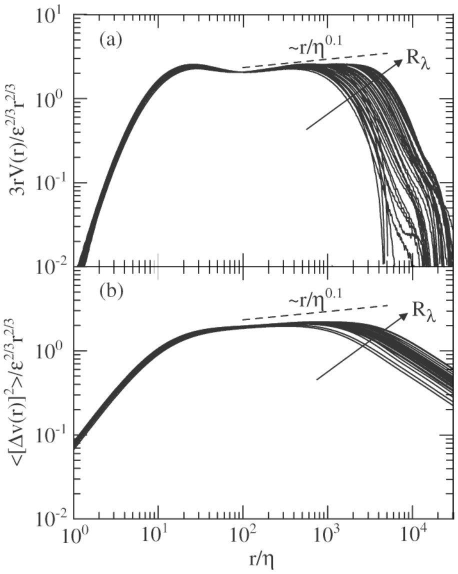

FIG. 3. (a) Compensated plots of $3rV(r)$ for grid turbulence ($R_\lambda = 400 \rightarrow 1200$). (b) The slope of $(r/\eta^{0.1})$ corresponds to that found in the direct numerical simulations of Kaneda *et al.* [8].

scales, $r \gg \eta$, viscous forces may be neglected and (12) integrates to give

$$\frac{d}{dt}\int_r^\infty V dr = -\Pi_V(r), \qquad r \gg \eta. \tag{14}$$

Since $\int_r^\infty V dr$ is a measure of the energy held in eddies whose size exceeds r, Π_V must represent the energy transferred by inertial forces from eddies of size r or greater to those of size r or less. Following [1] we refer to Π_V as the *real-space kinetic energy flux*. From (13) we see that it is the ensemble average of the third-order quantity,

$$\hat{\Pi}_V(x, x+r) = \tfrac{3}{2}(\Delta \mathrm{v})_\perp(\Delta \mathrm{v})(\partial u_y/\partial x) + \tfrac{3}{4}(\Delta \mathrm{v})_\perp^2(\partial u_x/\partial x) - \tfrac{3}{2}r^{-1}(\Delta \mathrm{v})^3,$$

which can be readily measured in direct numerical simulations or in the laboratory. Indeed the probability density function of $\hat{\Pi}_V$ could be measured and compared with the various probability density functions used in different intermittency models.

Let us now consider the equilibrium range, $r \ll \ell$. In this range the Karman-Howarth equation yields

$$\langle[\Delta \mathrm{v}]^3\rangle - 6\nu\frac{d}{dr}\langle[\Delta \mathrm{v}]^2\rangle = -\frac{4}{5}\varepsilon r.$$

If we substitute for $\langle[\Delta \mathrm{v}]^3\rangle$ in (12) and (13), and integrate, we find

$$\frac{d}{dt}\int_r^\infty V dr = -\varepsilon, \qquad r \ll \ell.$$

When combined with (14) this shows us that, in the inertial subrange, the real-space kinetic energy flux is $\Pi_V(r) = \varepsilon$, $\eta \ll r \ll \ell$, as expected. Moreover, it is readily confirmed

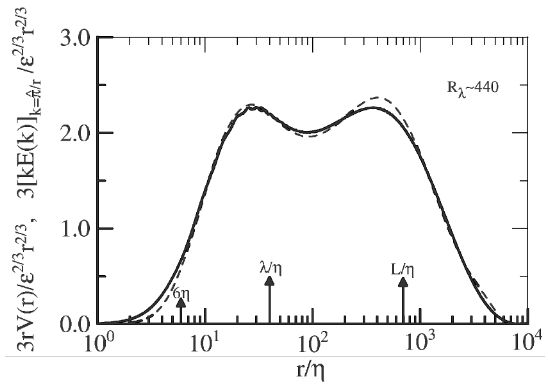

FIG. 4. Compensated plots of $3rV(r)$ (solid line) and $[3kE(k)]_{k=\hat{\pi}/r}$ (dashed line) for grid turbulence ($R_\lambda = 440$).

that the relationship between $\Pi_V(r)$ and $\Pi_E(k)$ is the same as that between $rV(r)$ and $kE(k)$. For example:

$$\Pi_V(r) = \frac{3\sqrt{\pi}}{2\sqrt{2}}\int_0^\infty k^{-1}\Pi_E(k)(rk)^{1/2}J_{7/2}(rk)dk. \tag{15}$$

It follows from (10) and Fig. 4 that $\Pi_V(r) \approx \Pi_E(k = \hat{\pi}/r)$, $\eta \leq r \leq \ell$. Thus $\Pi_V(r)$ offers us a way of quantifying the flux of energy across the different scales, without the need for Fourier analysis. One immediate benefit of using $V(r)$ and $\Pi_V(r)$ is that we can develop closure models in real space in the spirit of the spectral closures. Undoubtedly the simplest of these is Obukhov's constant skewness model for the equilibrium range. It is shown in [1] that this produces results remarkably close to the numerical and experimental data, reproducing phenomena such as the bottleneck, a feature which the equivalent spectral closures do not capture.

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